Приложение 3

Произведение выражений (2.74) и (2.75) {или (П.2.12) и (П.2.13)}

$$\psi(\xi') = -\sqrt{\frac{1}{4\pi\, l_2}}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right) \tag{П.3.1}$$

$$\psi^*(\xi') = -\sqrt{\frac{1}{4\pi\, l_2}}\left(\frac{e^{i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}+\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\right) \tag{П.3.2}$$

равно

$$p(\xi')=\psi(\xi')\psi^*(\xi')=\frac{1}{4\pi l_2}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right)\left(\frac{e^{i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}+\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\right) \tag{П.3.3}$$

Открывая большие скобки, попарно перемножим слагаемые

$$\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\frac{e^{i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}=\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{i(\pi n_1-\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}$$

$$\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}=\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2+\xi'/\eta\right)^2}$$

$$\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}=\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2+\xi'/\eta\right)^2}$$

$$\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}=\frac{\left(e^{-i(\pi n_1-\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}$$

Сложим получившиеся выражения

$$\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{i(\pi n_1-\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}+\frac{2\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2+\xi'/\eta\right)^2}+\frac{\left(e^{-i(\pi n_1-\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}$$

Переставляя слагаемые местами и суммируя, поучим

$$\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{i(\pi n_1-\xi' l_2/\eta)}-1\right)+\left(e^{-i(\pi n_1-\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}+\frac{2\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2+\xi'/\eta\right)^2} \tag{П.3.4}$$

Выполним вычисления

1. $\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{i(\pi n_1-\xi' l_2/\eta)}-1\right)=e^{i2\pi n_1}-e^{i(\pi n_1+\xi' l_2/\eta)}-e^{i(\pi n_1-\xi' l_2/\eta)}+1$

2. $\left(e^{-i(\pi n_1-\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)=e^{-i2\pi n_1}-e^{-i(\pi n_1-\xi' l_2/\eta)}-e^{-i(\pi n_1+\xi' l_2/\eta)}+1$

3. $\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)=e^{0}-e^{i(\pi n_1+\xi' l_2/\eta)}-e^{-i(\pi n_1+\xi' l_2/\eta)}+1=$

$$=1-(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})+1=-(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})+2=$$

$$-2[(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})/2-1]= \ -2[\cos(\pi n_1+\xi' l_2/\eta)-1] \qquad (\text{П.3.5})$$

где учтено выражение $\cos x=\dfrac{e^{ix}+e^{-ix}}{2}$.

Сложим 1 и 2

$$e^{i2\pi n_1}-e^{i(\pi n_1+\xi' l_2/\eta)}-e^{i(\pi n_1-\xi' l_2/\eta)}+e^{-i2\pi n_1}-e^{-i(\pi n_1-\xi' l_2/\eta)}-e^{-i(\pi n_1+\xi' l_2/\eta)}+2$$

Перегруппируем слагаемые

$$(e^{i2\pi n_1}+e^{-i2\pi n_1})-(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})-(e^{i(\pi n_1-\xi' l_2/\eta)}+e^{-i(\pi n_1-\xi' l_2/\eta)})+2$$

или

$$2[(e^{i2\pi n_1}+e^{-i2\pi n_1})/2-(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})/2-(e^{i(\pi n_1-\xi' l_2/\eta)}+e^{-i(\pi n_1-\xi' l_2/\eta)})/2+1]=$$

$$=\ 2[\cos 2\pi n_1-\cos(\pi n_1+\xi' l_2/\eta)-\cos(\pi n_1-\xi' l_2/\eta)+1] \qquad (\text{П.3.6})$$

Подставим слагаемые (П.3.5) и (П.3.6) в (П.3.4), получим

$$\frac{2[(\cos 2\pi n_1-\cos(\pi n_1+\xi' l_2/\eta)-\cos(\pi n_1-\xi' l_2/\eta)+1]}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}-\frac{4[\cos(\pi n_1+\xi' l_2/\eta)-1]}{(\pi n_1/l_2+\xi'/\eta)^2} \qquad (\text{П.3.7})$$

Теперь вставим (П.3.7) в (П.3.3)

$$p(\xi')=\frac{1}{4\pi l_2}\left(\frac{2[(\cos 2\pi n_1-\cos(\pi n_1+\xi' l_2/\eta)-\cos(\pi n_1-\xi' l_2/\eta)+1]}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}-\frac{4[\cos(\pi n_1+\xi' l_2/\eta)-1]}{(\pi n_1/l_2+\xi'/\eta)^2}\right)$$

(П.3.8)

Воспользуемся двумя тригонометрическими формулами

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad \text{и} \quad \cos x \cos y = \frac{1}{2}[\cos(x-y)+\cos(x+y)] \tag{П.3.9}$$

Откуда следует

$$\cos 2\pi n_1 + 1 = 2\cos^2 \pi n_1 \tag{П.3.10}$$

$$\cos(\pi n_1 - \xi' l_2/\eta) + \cos(\pi n_1 + \xi' l_2/\eta) = 2\cos(\pi n_1)\cos(\xi' l_2/\eta) \tag{П.3.11}$$

С учетом (П.3.10) и (П.3.11) выражение (П.3.8) принимает вид

$$p(\xi') = \frac{1}{4\pi l_2}\left(\frac{2[2\cos^2 \pi n_1 - 2\cos(\pi n_1)\cos(\xi' l_2/\eta)]}{\left(\pi n_1/l_2\right)^2 - \left(\xi'/\eta\right)^2} - \frac{4[\cos(\pi n_1 + \xi' l_2/\eta) - 1]}{\left(\pi n_1/l_2 + \xi'/\eta\right)^2}\right)$$

Выполняя упрощения

$$p(\xi') = \frac{1}{4\pi l_2}\left(\frac{4[\cos^2 \pi n_1 - \cos(\pi n_1)\cos(\xi' l_2/\eta)]}{\left(\pi n_1/l_2\right)^2 - \left(\xi'/\eta\right)^2} - \frac{4[\cos(\pi n_1 + \xi' l_2/\eta) - 1]}{\left(\pi n_1/l_2 + \xi'/\eta\right)^2}\right)$$

окончательно получим

$$p(\xi') = \frac{1}{\pi l_2}\left(\frac{[\cos^2 \pi n_1 - \cos(\pi n_1)\cos(\xi' l_2/\eta)]}{\left(\pi n_1/l_2\right)^2 - \left(\xi'/\eta\right)^2} - \frac{[\cos(\pi n_1 + \xi' l_2/\eta) - 1]}{\left(\pi n_1/l_2 + \xi'/\eta\right)^2}\right) \tag{П.3.12}$$