Приложение 2

Вычисление интегралов

Возьмем интегралы (2.73)

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \int_0^{l_2} \sqrt{\frac{2}{l_2}} \sin(\pi n_1 \xi / l_2) \exp\{i\xi'\xi/\eta\} d\xi\ , \qquad (\text{П.2.1})$$

$$\psi^*(\xi') = \frac{1}{\sqrt{2\pi}} \int_0^{l_2} \sqrt{\frac{2}{l_2}} \sin(\pi n_1 \xi / l_2) \exp\{-i\xi'\xi/\eta\} d\xi \qquad (\text{П.2.2})$$

Начнем с интеграла (П.2.1), и воспользуемся формулой $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ и представим (П.2.1) в виде

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_0^{l_2} \frac{e^{i\pi n_1 \xi / l_2} - e^{-i\pi n_1 \xi / l_2}}{2i} e^{i\xi'\xi/\eta} d\xi \qquad (\text{П.2.3})$$

Выполним последовательность следующих преобразований

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_0^{l_2} \frac{e^{i\pi n_1 \xi / l_2} e^{i\xi'\xi/\eta} - e^{-i\pi n_1 \xi / l_2} e^{i\xi'\xi/\eta}}{2i} d\xi$$

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_{-\infty}^{\infty} \frac{e^{i\pi n_1 \xi / l_2 + i\xi'\xi/\eta} - e^{-i\pi n_1 \xi / l_2 + i\xi'\xi/\eta}}{2i} d\xi$$

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_0^{l_2} \frac{e^{i\xi(\pi n_1 / l_2 + \xi'/\eta)} - e^{-i\xi(\pi n_1 / l_2 - \xi'/\eta)}}{2i} d\xi$$

$$\psi(\xi') = \frac{1}{2i\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_0^{l_2} e^{i\xi(\pi n_1 / l_2 + \xi'/\eta)} - e^{-i\xi(\pi n_1 / l_2 - \xi'/\eta)} d\xi$$

В результате данных преобразований, получаем

$$\psi(\xi') = \frac{1}{2i\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \left(\int_0^{l_2} e^{i\xi(\pi n_1 / l_2 + \xi'/\eta)} d\xi - \int_0^{l_2} e^{-i\xi(\pi n_1 / l_2 - \xi'/\eta)} d\xi \right) \qquad (\text{П.2.4})$$

Возьмем первый интеграл в (П.2.4)

$$\int_{-\infty}^{\infty} e^{i\xi(\pi n_1 \xi/l_2+\xi'/\eta)}d\xi = \frac{\int_0^{l_2} e^{i\xi(\pi n_1/l_2+\xi'/\eta)}d[i\xi(\pi n_1/l_2+\xi'/\eta)]}{i(\pi n_1/l_2+\xi'/\eta)}$$

$$\frac{\int_0^{l_2} e^{i\xi(\pi n_1 \xi/l_2+\xi'/\eta)}di\xi(\pi n_1/l_2+\xi'/\eta)}{i(\pi n_1/l_2+\xi'/\eta)} = \left.\frac{e^{i\xi(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)} - \frac{e^{i0(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)} - \frac{1}{i(\pi n_1/l_2+\xi'/\eta)}$$

В результате данных вычислений, получаем

$$\left.\frac{e^{i\xi(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2+\xi'/\eta)}-1}{i(\pi n_1/l_2+\xi'/\eta)} \qquad (П.2.5)$$

Возьмем второй интеграл в (П.2.4)

$$\int_0^{l_2} e^{-i\xi(\pi n_1 \xi/l_2-\xi'/\eta)}d\xi = \frac{\int_0^{l_2} e^{-i\xi(\pi n_1/l_2-\xi'/\eta)}d[-i\xi(\pi n_1/l_2-\xi'/\eta)]}{-i(\pi n_1/l_2-\xi'/\eta)}$$

$$\frac{\int_0^{l_2} e^{-i\xi(\pi n_1/l_2-\xi'/\eta)}d-i\xi(\pi n_1/l_2-\xi'/\eta)}{-i(\pi n_1/l_2-\xi'/\eta)} = \left.\frac{e^{-i\xi(\pi n_1 \xi/l_2-\xi'/\eta)}}{-i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2}$$

$$\left.\frac{e^{-i\xi(\pi n_1/l_2-\xi'/\eta)}}{-i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{-il_2(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)} - \frac{1}{i(\pi n_1/l_2-\xi'/\eta)}$$

В результате данных вычислений, получаем

$$\left.\frac{e^{-i\xi(\pi n_1/l_2-\xi'/\eta)}}{-i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{-il_2(\pi n_1/l_2-\xi'/\eta)}-1}{-i(\pi n_1/l_2-\xi'/\eta)} \qquad (П.2.6)$$

Подставляя (П.2.5) и (П.2.6) и (П.2.4), имеем

$$\psi(\xi') = \frac{1}{2i\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{il_2(\pi n_1/l_2+\xi'/\eta)}-1}{i(\pi n_1/l_2+\xi'/\eta)} - \frac{e^{-il_2(\pi n_1/l_2-\xi'/\eta)}-1}{-i(\pi n_1/l_2-\xi'/\eta)}\right)$$

Выполним преобразования

$$\psi(\xi') = \frac{1}{2ii\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right)$$

$$\psi(\xi') = -\frac{1}{2\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right)$$

Окончательно получаем результат интегрирования (П.2.1)

$$\psi(\xi') = -\sqrt{\frac{1}{4\pi\, l_2}}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right) \qquad (\text{П.2.7})$$

Аналогично возьмем интеграл (П.2.2)

$$\psi^*(\xi') = \frac{1}{\sqrt{2\pi}}\int_0^{l_2}\sqrt{\frac{2}{l_2}}\sin(\pi n_1\xi/l_2)\exp\{-i\xi'\xi/\eta\}d\xi$$

Представим (П.2.2) в виде

$$\psi^*(\xi') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_0^{l_2}\frac{e^{i\pi n_1\xi/l_2}-e^{-i\pi n_1\xi/l_2}}{2i}e^{-i\xi'\xi/\eta}d\xi$$

$$\psi^*(\xi') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_0^{l_2}\frac{e^{i\pi n_1\xi/l_2}e^{-i\xi'\xi/\eta}-e^{-i\pi n_1\xi/l_2}e^{-i\xi'\xi/\eta}}{2i}d\xi$$

$$\psi^*(\xi') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_{-\infty}^{\infty}\frac{e^{i\pi n_1\xi/l_2-i\xi'\xi/\eta}-e^{-i\pi n_1\xi/l_2-i\xi'\xi/\eta}}{2i}d\xi$$

$$\psi^*(\xi') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_0^{l_2}\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}-e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}}{2i}d\xi$$

$$\psi^*(\xi') = \frac{1}{2i\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_0^{l_2}e^{i\xi(\pi n_1/l_2-\xi'/\eta)}-e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}d\xi$$

В результате данных преобразований, получаем

$$\psi^*(\xi') = \frac{1}{2i\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\int_0^{l_2}e^{i\xi(\pi n_1/l_2-\xi'/\eta)}d\xi-\int_0^{l_2}e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}d\xi\right) \qquad (\text{П.2.8})$$

Возьмем первый интеграл в (П.2.8)

$$\int_{-\infty}^{\infty} e^{i\xi(\pi n_1\xi/l_2+\xi'/\eta)}d\xi = \frac{\int_0^{l_2} e^{i\xi(\pi n_1/l_2-\xi'/\eta)}d[i\xi(\pi n_1/l_2-\xi'/\eta)]}{i(\pi n_1\xi/l_2-\xi'/\eta)}$$

$$\frac{\int_0^{l_2} e^{i\xi(\pi n_1\xi/l_2-\xi'/\eta)}di\xi(\pi n_1/l_2-\xi'/\eta)}{i(\pi n_1/l_2-\xi'/\eta)} = \left.\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)} - \frac{e^{i0(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)} - \frac{1}{i(\pi n_1/l_2-\xi'/\eta)}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2-\xi'/\eta)}-1}{i(\pi n_1/l_2-\xi'/\eta)} \qquad (\text{П.2.9})$$

Возьмем второй интеграл в (П.2.8)

$$\int_0^{l_2} e^{-i\xi(\pi n_1\xi/l_2+\xi'/\eta)}d\xi = \frac{\int_0^{l_2} e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}d[-i\xi(\pi n_1/l_2+\xi'/\eta)]}{-i(\pi n_1/l_2+\xi'/\eta)}$$

$$\frac{\int_0^{l_2} e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}d-i\xi(\pi n_1/l_2+\xi'/\eta)}{-i(\pi n_1/l_2+\xi'/\eta)} = \left.\frac{e^{-i\xi(\pi n_1\xi/l_2+\xi'/\eta)}}{-i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2}$$

$$\left.\frac{e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}}{-i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2} = \frac{e^{-il_2(\pi n_1/l_2+\xi'/\eta)}-1}{-i(\pi n_1/l_2+\xi'/\eta)} \qquad (\text{П.2.10})$$

Подставляя (П.2.5) и (П.2.6) и (П.2.4), имеем

$$\psi^*(\xi') = \frac{1}{2i\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{il_2(\pi n_1/l_2-\xi'/\eta)}-1}{i(\pi n_1/l_2-\xi'/\eta)} - \frac{e^{-il_2(\pi n_1/l_2+\xi'/\eta)}-1}{-i(\pi n_1/l_2+\xi'/\eta)}\right)$$

Выполним преобразования

$$\psi^*(\xi') = \frac{1}{2ii\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)} + \frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\right)$$

$$\psi^*(\xi') = -\frac{1}{2\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{i(\pi n_1 - \xi' l_2/\eta)} - 1}{(\pi n_1/l_2 - \xi'/\eta)} + \frac{e^{-i(\pi n_1 + \xi' l_2/\eta)} - 1}{(\pi n_1/l_2 + \xi'/\eta)}\right)$$

Окончательно получаем результат интегрирования (П.2.2)

$$\psi^*(\xi') = -\sqrt{\frac{1}{4\pi\, l_2}}\left(\frac{e^{i(\pi n_1 - \xi' l_2/\eta)} - 1}{(\pi n_1/l_2 - \xi'/\eta)} + \frac{e^{-i(\pi n_1 + \xi' l_2/\eta)} - 1}{(\pi n_1/l_2 + \xi'/\eta)}\right) \qquad (\text{П}.2.11)$$

Итак, результатами взятия интегралов (П.2.1) и (П.2.2) являются выражения (П.2.7) и (П.2.11):

$$\psi(\xi') = -\sqrt{\frac{1}{4\pi\, l_2}}\left(\frac{e^{i(\pi n_1 + \xi' l_2/\eta)} - 1}{(\pi n_1/l_2 + \xi'/\eta)} + \frac{e^{-i(\pi n_1 - \xi' l_2/\eta)} - 1}{(\pi n_1/l_2 - \xi'/\eta)}\right) \qquad (\text{П}.2.12)$$

$$\psi^*(\xi') = -\sqrt{\frac{1}{4\pi\, l_2}}\left(\frac{e^{i(\pi n_1 - \xi' l_2/\eta)} - 1}{(\pi n_1/l_2 - \xi'/\eta)} + \frac{e^{-i(\pi n_1 + \xi' l_2/\eta)} - 1}{(\pi n_1/l_2 + \xi'/\eta)}\right) \qquad (\text{П}.2.13)$$