12 Development of the Stochastic Interpretation of Quantum Mechanics by E. Nelson. Derivation of the Schrödinger-Euler-Poisson Equations

Mikhail Batanov-Gaukhman
Ph.D., Associate Professor, Institute No. 2 "Aircraft, rocket engines and power plants",
Federal State Budgetary Educational Institution of Higher Education "Moscow Aviation Institute (National Research University)", Volokolamsk highway 4, Moscow, Russian Federation (alsignat@yandex.ru)

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Abstract: The aim of the article is to develop a stochastic interpretation of the quantum mechanics by E. Nelson. Based on the consideration of the averaged states of a chaotically wandering particle, the stationary and time-dependent stochastic Schrödinger-Euler-Poisson equations (47) and (92) were obtained, which coincided with the corresponding Schrödinger equations up to coefficients. In this case, the ratio of the reduced Planck constant to the particle mass is expressed through the averaged characteristics of a three-dimensional random process in which the considered wandering particle participates. The obtained stochastic equations (39), (47), (88), (92) are suitable for describing quantum phenomena and averaged states of particles not only at atomic and subatomic scales, but also similar stochastic systems of the micro- and macroworld.

Keywords: Stochastic process, Schrödinger equation, Planck constant, derivation of the Schrödinger equation. stochastic equation, stochastic quantum mechanics

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List of abbreviations and definitions

MSQM is the massless stochastic quantum mechanics;
QM is the quantum mechanics;
TME is the total mechanical energiality;
PSRP is the pseudo-stationary random process;
SQM is the stochastic quantum mechanics;
SRP is the stationary random process;
PA is the probability amplitude;
PDF is the probability density function;
ChWP is the chaotically wandering particle;
Pico-particle is a particle or antiparticle with a size of ~ 10^{-10} – 10^{-13} cm;
Micro-particle is a particle with dimensions of ~ 10^{-7} – 10^{-3} cm;
Macro-particle is a compact bodies with dimensions of ~ 10^{-2} – 10^{4} cm;
Mega-particle is a planet and other cosmic bodies with sizes of ~ 10^{5} – 10^{9} cm.
$s = S/m$ is "efficiency" of the particle with mass m;
$\varepsilon = E/m$ is "total mechanical energiality" of the particle with mass m;
$u = U/m$ is "potential energiality" of the particle with mass m;
$t(v_x) = T(v_x)/m$ is "kinetic energiality" of the particle with mass m.

1 BACKGROUND AND INTRODUCTION

In modern physics, there are several dozen interpretations of Quantum Mechanics (QM). Each of them has its own advantages and disadvantages, but none of them is precisely defined, since many researchers often put different meanings into the same concepts.

One of the reasons for this situation in quantum physics is associated with a different attitude to the wave function $\Psi(x,t)$.

Most experts agree with M. Born's statement that the square of the modulus of the wave function of a particle $\Psi(x,t)$ is equal to the probability density function (PDF) of the particle's location at a point x

$$|\Psi(x,t)|^2 = \rho(x,t).$$

However, it should be borne in mind that, in general, this PDF is a complex function of several factors associated with the measurement process.

$$|\Psi(x,t)|^2 = \rho(x,t) = f\,[\rho_p(x,t), \rho_m(x,t), \rho_e(x,t), \rho_d(x,t), \rho_o(x,t)], \tag{1}$$

where

$\rho_p(x,t)$ is the PDF associated with the chaotic behavior of the particle;

$\rho_m(x,t)$ is the PDF, associated with the method errors;

$\rho_e(x,t)$ is the PDF, associated with the influence of the external environment;

$\rho_d(x,t)$ is the PDF, associated with the instrument errors;

$\rho_o(x,t)$ is the PDF, associated with the operator errors.

An example of functional dependence (1) is the PDF

$$|\Psi(x,t)|^2 = \rho(x,t) = \frac{1}{\sqrt{2\pi[\sigma_{px}^2 + \sigma_{mx}^2 + \sigma_{ex}^2(t) + \sigma_{dx}^2 + \sigma_{ox}^2]}} \exp\left\{ -\frac{x^2}{2[\sigma_{px}^2 + \sigma_{mx}^2 + \sigma_{ex}^2(t) + \sigma_{dx}^2 + \sigma_{ox}^2]} \right\}$$

where (2)

$$\sigma_{ix}^2 = \int_{-\infty}^{+\infty} \rho_i(x) x^2 dx \ \text{(here } i = p,m,d,o\text{)}; \qquad \sigma_{ex}^2(t) = \int_{-\infty}^{+\infty} \rho_e(x,t) x^2 dx$$

is a variance of the i-th influencing factor on the measurement result.

All of the above factors are present when measuring the physical characteristics of particles of any scale. However, depending on the particle size, these factors affect the result differently.

At the same time, almost all specialists who study the properties of non-relativistic pico-particles (i.e., particles with characteristic sizes of atomic and subatomic scales, $10^{-10} - 10^{-13}$ cm) use the same mathematical apparatus of quantum mechanics (QM), designed to predict possible configurations and evolutions of wave functions $\Psi(x,t)$ of one particle or an ensemble of identical particles.

Focusing on certain factors influencing the measurement process, using the same mathematical apparatus, leads to the development of different interpretations of QM.

For example, in a number of experiments pico-particles are extremely sensitive to the influence of the measuring system and the observer on them. In this case, the wave function (1) should take into account all influencing factors, while the methodology for perceiving the results obtained is most consistent with the Copenhagen interpretation of QM, formulated by Niels Bohr and Werner Heisenberg.

In other experiments, the factors that interfere with the measurement are so insignificant that they can be neglected. For example, we judge the possible states of an electron in a hydrogen atom by its emission spectrum. If we abstract from the slight broadening of the spectral lines associated with the influence of various accompanying factors, then in this case PDF (1) takes the form

$$|\Psi(x,t)|^2 = \rho(x,t) = f\,[\rho_p(x,t)\,\rho_e(x,t)].$$

This wave function characterizes only the properties of the electron itself, taking into account the influence of the vacuum, leading to the Lamb shift of the spectral lines.

In this article, we will adhere to the Stochastic interpretation of quantum mechanics, most clearly expressed in the works of Edward Nelson [2,3,4], published in 1966 – 1985.

In addition to E. Nelson, this interpretation of QM was developed by R. Fürth [5], I. Fényes [6], W. Weizel [7], M. Pavon, [8]. An alternative stochastic interpretation of QM was developed by R. Tsekov [9].

Nelson's stochastic interpretation is associated with the logical construction of QM by analogy with the theory of Brownian motion [more precisely, the Ornstein-Uhlenbeck process].

In Nelson's interpretation, the reason for the chaotic behavior of a pico-particle is associated with the effect of vacuum fluctuations on it. The diffusion coefficient of such a stochastic process turns out to be imaginary due to the absence of friction and the specificity of the vacuum viscosity. Thus, in the stochastic interpretation of QM, the primary is not the wave function $\Psi(x,t)$, but complex small-scale curvatures of the space-time continuum (i.e., the Wheeler-Bohm-Vigier "quantum foam"), which affect to the colloidal pico-particle.

In this case, PDF (1) takes the simplest form

$$|\Psi(x,t)|^2 = \rho(x,t) = \rho_e(x,t) = \psi(x,t)\,\psi^*(x,t), \tag{3}$$

since it characterizes only the chaotic behavior of a structureless particle under the influence of a turbulent environment.

Recall that the Langevin and Fokker-Planck stochastic equations describe Brownian motion without taking into account the structure of particles and the uncertainty associated with measurement errors. However, there is a fundamental difference between Brownian particles (~ 10^{-4}cm in size) and pico-particles (~ 10^{-10} – 10^{-13}cm in size). Brownian (colloidal) particles can be observed with a microscope, practically without affecting them, while pico-particles, in principle, cannot be observed directly.

In this article, the maximally simplified (more precisely, not taking into account the measurement error and the influence of other particles) probability amplitude (PA) $\Psi(x,t) = \psi(x,t)$ will be called the "pure" wave function.

It should be noted that in most books and textbooks on quantum mechanics, initially the PA $\Psi(x,t)$ means the "pure" wave function of the particle. This is one of the reasons for the lack of agreement between theorists and experimenters, as well as between specialists working in various fields of quantum physics. Apparently, it was the attitude to the "pure" or "impure" wave function $\Psi(x,t)$ that caused the disputes between A. Einstein (who studied Brownian motion in his youth) and N. Bohr (whose early works were associated with the atomic emission spectra).

So, in this article, under the stochastic interpretation of quantum mechanics by Edward Nelson, we mean a version of QM in which the wave function $\psi(x,t)$ characterizes only the chaotic behavior of a wandering particle under the influence of environmental fluctuations, without taking into account measurement errors and the influence of the operator. This particle (like a Brownian corpuscle) has a volume and a chaotic trajectory of motion. In this case, the wave function $\psi(x,t)$ has the statistical character attributed to it by M. Born.

At the same time, it is taken into account that within the framework of the Nelson's stochastic interpretation of the QM, the "pure" PA $\psi(x,t)$ turns out to be a kind of "intellectual thing-in-itself". This is because the "pure" wave function $\psi(x,t)$ can be found out only by solving stochastic differential equations. Any attempt to perform a measurement will lead to a partial distortion or complete destruction of the stochastic system under study, and hence to a change in its PA $\psi(x,t)$.

This article attempts to develop the foundations of massless stochastic quantum mechanics (MSQM), which is a development of Nelson's stochastic quantum mechanics (SQM) [2,3], and proposes a solution to the problem of measuring "pure" parameters of stochastic quantum systems.

A probabilistic model of a chaotically wandering particle (ChWP) is considered, which, like the pico-particle by E. Nelson [2], has a volume and a continuous trajectory of motion. But in contrast to the SQM [2], in the MSQM there are no restrictions on the size of the investigated particle. Based on this model (by a method different from [2]), stochastic equations (39) (47), (89) and (92) are derived, which are a generalization of the Schrödinger equations.

According to the author, the main advantage of stochastic equations (39) (47), (89), and (92) obtained in this article is that they are suitable for describing averaged states and calculating quantum parameters of a chaotically wandering particle (ChWP) of any scale.

In other words, the MSQM, proposed in this article, can make it possible to study stochastic processes not only at the atomic and subatomic levels of the organization of matter, but also at the macrolevel. For example, one can estimate the probability of manifestation of quantum effects when averaging the chaotic displacements of the singularity in the galactic nucleus and predict the consequences arising from this.

2 METHOD

2.1 Probabilistic model of a wandering particle

Consider a particle that, under the action of a multitude of unrelated force factors (including a fluctuating environment), constantly wanders chaotically in the vicinity of a conditional center combined with the origin of a fixed coordinate system XYZ (Figure 1).

Examples of such a chaotic movement of a particle are: an electron in a hydrogen atom, vibrations of an atom in a crystal lattice, chaotic flights of a fly in a jar, trembling of the nucleus of a biological cell, moving the center of mass of an embryo in the womb, wandering the tip of a tree branch under gusts of wind, moving the center of mass of a school fish in the ocean, vibration and displacement of the iron core in the bowels of the planet, etc.

The total mechanical energy of a chaotically wandering particle (ChWP) at each moment of time and at each point in 3-dimensional space is equal to

$$E(x,y,z,t) = T(x,y,z,t) + U(x,y,z,t), \tag{4}$$

where $T(x,y,z,t)$ is the kinetic energy of the particle due to its velocity at time t;

$U(x,y,z,t)$ is the potential energy of the particle associated with the elasticity of its environment, which tends to return this particle to the conditional center.

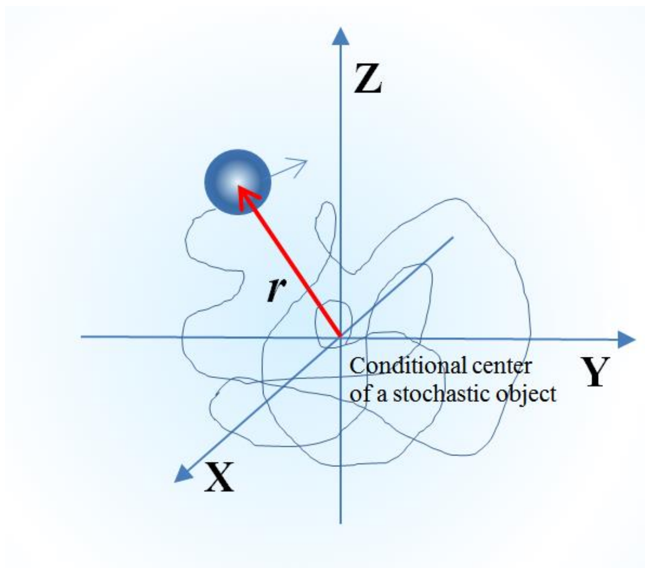


Fig. 1. A simplified model of a particle that randomly wanders in the vicinity of the conditional center, combined with the origin of the coordinate system XYZ

In the general case, each of the ChWP energies: $E(x,y,z,t)$, $T(x,y,z,t)$ and $U(x,y,z,t)$ is a random function of time and of a place relative to the conditional center. But the energies $T(x,y,z,t)$ and $U(x,y,z,t)$ so smoothly pass into each other that their sum is always equal to $E(x,y,z,t)$, i.e. condition (2) is always fulfilled.

If the ChWP speed is small compared to the speed of light, then according to nonrelativistic mechanics, it has kinetic energy

$$T(x,y,z,t) = \frac{p_x^2(x,y,z,t) + p_y^2(x,y,z,t) + p_z^2(x,y,z,t)}{2m},$$

or in a compact form

$$T(\vec{r},t) = \frac{p_x^2(\vec{r},t) + p_y^2(\vec{r},t) + p_z^2(\vec{r},t)}{2m},$$

where $p_x(\vec{r},t)$, $p_y(\vec{r},t)$, $p_z(\vec{r},t)$ are the instantaneous values of the components of the momentum vector of a wandering particle at time t at a point with coordinates x,y,z; m is the mass of the particle; $\vec{r}$ is the radius vector with the origin in the conditional center of the stochastic system under study ($r^2 = x^2 + y^2 + z^2$) (Figure 1).

The type of potential energy of the particle $U(x,y,z,t)$ at this stage of the study is not specified.

To simplify the mathematical calculations, consider a one-dimensional case that does not limit the generality of conclusions. In the case of three dimensions, only the number of integrations increases.

The action of a moving particle in nonrelativistic mechanics is defined as follows [10]

$$S(t) = \int_{t_1}^{t_2} [T(p_x,t) - U(x,t)]dt + E(x,t)t. \tag{5}$$

Due to the complexity of the trajectory of the ChWP movement, we will be interested not in the action (5) itself, but in its averaging

$$\bar{S}(t) = \int_{t_1}^{t_2} [\overline{T(p_x,t)} - \overline{U(x,t)}]dt + \overline{E(x,t)}\Delta t. \tag{6}$$

The averaging of the action (6) is carried out over realizations taken over the same time interval $\Delta t = t_2 - t_1$.

Let's represent the average kinetic energy of the ChWP in the form

$$\overline{T(p_x,t)} = \frac{1}{2m}\int_{-\infty}^{\infty} \rho(p_x,t)\, p_x^2 dp_x, \tag{7}$$

where $\rho(p_x,t)$ is the PDF of the x-component of the particle momentum p_x.

In (7), averaging occurs over all possible p_x of a particle, independent of time and of its position in the considered region of 3-dimensional space (see Figure 1).

Let’s represent the averaged potential energy and the averaged total mechanical energy of the ChWP in the form

$$\overline{U(x,t)} = \int_{-\infty}^{\infty} \rho(x,t) U(x,t) dx, \qquad (8)$$

$$\overline{E(x,t)} = \int_{-\infty}^{\infty} \rho(x,t) E(x,t) dx, \qquad (9)$$

where $\rho(x,t)$ is the PDF of the possible location of the projection of a wandering particle on the X axis (see Figure 1 a,b) at time t.

Substituting Ex.s (7), (8) and (9) in (6), we obtain the average action of a chaotically wandering particle (ChWP)

$$\bar{S}_x(t) = \int_{t_1}^{t_2} \left\{ \frac{1}{2m} \int_{-\infty}^{\infty} \rho(p_x,t) p_x^2 dp_x - \int_{-\infty}^{\infty} \rho(x,t) U(x,t) dx + \int_{-\infty}^{\infty} \rho(x,t) E(x,t) dx \right\} dt. \qquad (10)$$

or $$\bar{S}_x(t) = \int_{t_1}^{t_2} \left\{ \frac{1}{2m} \int_{-\infty}^{\infty} \rho(p_x,t) p_x^2 dp_x + \int_{-\infty}^{\infty} \rho(x,t) [E(x,t) - U(x,t)] dx \right\} dt. \qquad (11)$$

The average action of the considered "stochastic system" (11) provides for the most difficult case, when various variants of the change in the averaged state of the ChWP with time t are possible.

2.2 Stationary state of ChWP

Let's consider a stationary version of a stochastic system, when the average behavior of ChWP does not depend on time.

In this case, the behavior of the ChWP is described by a stationary random process (SRP) (see Appendix 1), whereby none of its averaged characteristics depend on time:

$$\rho(p_x,t) = \rho(p_x)$$

$$\overline{T(p_x,t)} = \frac{1}{2m} \int_{-\infty}^{\infty} \rho(p_x) p_x^2 dp_x, \qquad (12)$$

$$\overline{U(x,t)} = \int_{-\infty}^{\infty} \rho(x)\overline{U(x)}dx, \tag{13}$$

$$\overline{E(x,t)} = \int_{-\infty}^{\infty} \rho(x)\overline{E(x)}dx, \tag{14}$$

where $\overline{U(x)}$ is the average potential energy of the ChWP at the point x;

$\overline{E(x)}$ is the average total mechanical energy of the ChWP at the point x.

Substituting (12) – (14) into the averaged action (6), we obtain

$$\bar{S}_x = \int_{t_1}^{t_2} \left\{ \frac{1}{2m} \int_{-\infty}^{\infty} \rho(p_x) p_x^2 dp_x - \int_{-\infty}^{\infty} \rho(x)\overline{U(x)}dx + \int_{-\infty}^{\infty} \rho(x)\overline{E(x)}dx \right\} dt. \tag{15}$$

Let's represent the average action (15) in coordinate form. To do this, perform the following steps:

1. Let's write the PDF $\rho(x)$ in the form of the product of two probability amplitude (PA) $\psi(x)$:

$$\rho(x) = \psi(x)\,\psi(x). \tag{16}$$

2. Divide both sides of Ex. (15) by the particle mass m and change the variable $p_x = mv_x$ with the transformation Jacobian $J = 1/m$, as a result we get

$$\overline{s_x} = \int_{t_1}^{t_2} \left\{ \frac{1}{2} \int_{-\infty}^{\infty} \rho(v_x) v_x^2 dv_x - \int_{-\infty}^{\infty} \rho(x)\overline{u(x)}dx + \int_{-\infty}^{\infty} \rho(x)\overline{\varepsilon(x)}dx \right\} dt. \tag{17}$$

where v_x is the instantaneous value of the x-component of the ChWP velocity vector; s, $\varepsilon(x)$, $u(x)$, $t(v_x)$ – massless quantities, which we will assign the following names:

$$s = S/m \tag{18}$$

is the "efficiency" of the particle;

$$\varepsilon(x) = E(x)/m \tag{19}$$

is the "total mechanical energiality" of the particle;

$$u(x) = U(x)/m \tag{20}$$

is the "potential energiality" of the particle;

$$t(v_x) = T(v_x)/m = v_x{}^2/2 \tag{21}$$

is the "kinetic energiality" of the particle.

3. Let's use the coordinate representation of the averaged *x*-component of the velocity vector of the ChWP, raised to the *n*-th power (A2.1) (Appendix 2). In particular, for $n = 2$, we have

$$\overline{v_x^2} = \int_{-\infty}^{+\infty} \rho(v_x) v_x^n dv_x = \int_{-\infty}^{+\infty} \psi(x)\left(-i\eta_x \frac{\partial}{\partial x}\right)^2 \psi(x) dx, \tag{22}$$

where, according to A1.52, (see Appendix 1);

$$\eta_x = \frac{2\sigma_x^2}{\tau_{x\,кор}} \quad \text{is scale parameter,} \tag{23}$$

where σ_x is the standard deviation of the stationary random process $x(t)$ associated with the projection of the position of the ChWP on the X axis;

$\tau_{x\ cor}$ is the autocorrelation interval of the given random process $x(t)$.

4. Using Ex.s (12), (21) and (22), we represent the averaged kinetic energiality of the ChWP in the form

$$\overline{t(x,t)} = \frac{\overline{T(x,t)}}{m} = \frac{1}{2}\overline{v_x^2(x,t)} = \frac{1}{2}\int_{-\infty}^{\infty} \rho(v_x) v_x^2 dv_x = \frac{\eta_x^2}{2}\int_{-\infty}^{\infty} \psi(x)\frac{\partial^2 \psi(x)}{\partial x^2} dx. \tag{24}$$

5. The average potential energiality of a particle according to (13) and (20), and its average total mechanical energiality according to (14) and (19), taking into account (16), take the form

$$\overline{u(x,t)} = \frac{\overline{U(x,t)}}{m} = \int_{-\infty}^{\infty} \rho(x)\overline{u(x)}dx = \int_{-\infty}^{\infty} \psi(x)\overline{u(x)}\psi(x)dx, \tag{25}$$

$$\overline{\varepsilon(x,t)} = \frac{\overline{E(x,t)}}{m} = \int_{-\infty}^{\infty} \rho(x)\overline{\varepsilon(x)}dx = \int_{-\infty}^{\infty} \psi(x)\overline{\varepsilon(x)}\psi(x)dx. \tag{26}$$

6. Substitute Ex.s (24), (25) and (26) into the integral (17)

$$\overline{s_x} = \int_{t_1}^{t_2}\left\{\frac{\eta_x^2}{2}\int_{-\infty}^{\infty}\psi(x)\frac{\partial^2\psi(x)}{\partial x^2}dx - \int_{-\infty}^{\infty}\psi(x)\overline{u(x)}\psi(x)dx + \int_{-\infty}^{\infty}\psi(x)\overline{\varepsilon(x)}\psi(x)dx\right\}dt. \tag{27}$$

Combining all the terms in (27) under one integral, we obtain the following form of the average "efficiency" of the ChWP in the coordinate representation

$$\overline{s_x} = \int_{t_1}^{t_2}\int_{-\infty}^{\infty}\left(\frac{\eta_x^2}{2}\psi(x)\frac{\partial^2\psi(x)}{\partial x^2} - \psi^2(x,t)[\overline{\varepsilon(x)} - \overline{u(x)}]\right)dxdt. \qquad (28)$$

The average efficiency (28) characterizes the considered stochastic system in a stationary state, i.e., when the averaged state of a chaotically wandering particle (ChWP) does not change with time.

In this case, the energiality balance of the considered stochastic system can be represented as

$$\overline{\varepsilon(x,y,z)} = \overline{t(x,y,z)} + \overline{u(x,y,z)} = const. \qquad (28a)$$

Condition (28a) indicates that the ChWP, on average, does not lose its total mechanical energiality. In this case, it can be assumed that the viscosity of the medium surrounding the ChWP is complex, since the seething medium either takes a part of the total energiality from the particle, then returns it the same part of the energiality.

2.3 Stationary stochastic Euler-Poisson equation

Let's find the extremal of the functional (28), i.e., define the function $\psi(x)$ for which the average efficiency

$$\overline{s_x} = \int_{t_1}^{t_2}\int_{-\infty}^{\infty}\left(\frac{\eta_x^2}{2}\psi(x)\frac{\partial^2\psi(x)}{\partial x^2} + \psi^2(x)[\overline{\varepsilon(x)} - \overline{u(x)}]\right)dxdt. \qquad (29)$$

takes an extreme value.

Since there are no time-dependent functions in (29), we will seek the extremality condition of the internal integral

$$w = \int_{-\infty}^{\infty}\left(\frac{\eta_x^2}{2}\psi(x)\frac{\partial^2\psi(x)}{\partial x^2} + \psi^2(x)[\overline{\varepsilon(x)} - \overline{u(x)}]\right)dx \qquad (30)$$

In the calculus of variations, it was shown [21] that the extremal $y = f(x)$ of a general functional

$$N = \int_{-\infty}^{\infty} F\left(x, y, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}, ..., \frac{\partial^n y}{\partial x^n} \right) dx, \tag{31}$$

is determined by the Euler-Poisson equation [21]

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + ... + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0. \tag{32}$$

In the case of searching for the extremal of functional (30), we have

$$y = \psi(x), \quad F = \frac{\eta_x^2}{2} \psi(x) \frac{\partial^2 \psi(x)}{\partial x^2} + \psi^2(x)[\overline{\varepsilon(x)} - \overline{u(x)}], \tag{33}$$

in this case, the Euler-Poisson equation (32) is simplified

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0. \tag{34}$$

where

$$F_y = \frac{\eta_x^2}{2} \frac{\partial^2 \psi(x)}{\partial x^2} + 2\psi(x)[\overline{\varepsilon(x)} - \overline{u(x)}], \quad F_{y'} = 0 \quad \text{and} \quad F_{y''} = \frac{\eta_x^2}{2} \psi(x). \tag{35}$$

Substituting (35) into (34), taking into account

$$\frac{d^2 \psi(x)}{dx^2} = \frac{\partial^2 \psi(x)}{\partial x^2}, \tag{36}$$

we obtain the one-dimensional stochastic Euler-Poisson equation

$$\frac{\eta_x^2}{2} \frac{\partial^2 \psi(x)}{\partial x^2} + \psi(x)[\overline{\varepsilon(x)} - \overline{u(x)}] = 0. \tag{37}$$

Generalization to three dimensions leads to an increase in the number of integrations, while instead of Eq. (37), we get (38)

$$\frac{\eta_{x,y,z}^2}{2} \left\{ \frac{\partial^2 \psi(x,y,z)}{\partial x^2} + \frac{\partial^2 \psi(x,y,z)}{\partial y^2} + \frac{\partial^2 \psi(x,y,z)}{\partial z^2} \right\} + [\overline{\varepsilon(x,y,z)} - \overline{u(x,y,z)}]\, \psi(x,y,z) = 0$$

or in a compact form

$$\frac{\eta_r^2}{2} \nabla^2 \psi(\vec{r}) + [\overline{\varepsilon(\vec{r})} - \overline{u(\vec{r})}]\, \psi(\vec{r}) = 0, \tag{39}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator; (40)

$$\eta_r = \eta_{x,y,z} = \frac{2\sigma_r^2}{\tau_{r\,cor}} \tag{41}$$

is the scale parameter, where for the three-dimensional case:

$$\sigma_r = \frac{1}{\sqrt{3}}\sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \tag{42}$$

is the standard deviation of a random 3-dimensional trajectory of the ChWP from the conditional center of the stochastic system under consideration (Figure 1);

$$\tau_{r\,cor} = \frac{1}{3}\left(\tau_{x\,cor} + \tau_{y\,cor} + \tau_{z\,cor}\right) \tag{43}$$

is the autocorrelation interval of a given 3-dimensional stationary random process.

Ex. (39) will be called *the massless stationary (i.e., time - independent) stochastic Euler-Poisson equation* for finding the extremal $\psi(\bar{r}) = \psi(x,y,z)$ of functional (44)

$$\overline{s_r} = \int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\eta_r^2}{2}\psi(x,y,z)\nabla^2\psi(x,y,z) + \psi^2(x,y,z)[\overline{\varepsilon(x,y,z)} - \overline{u(x,y,z)}]\right)dxdydzdt,$$

with the energiality balance of the considered stationary stochastic system (28a).

2.4 Stochastic Schrödinger-Euler-Poisson equation

Suppose that the total mechanical energiality of the ChWP always remains constant, then equation (4), taking into account Ex.s (19) – (21), is simplified to

$$\varepsilon = t(x,y,z,t) + u(x,y,z,t) = const. \tag{45}$$

In such a stationary stochastic system, the kinetic energiality of the ChWP $t(x,y,z,t)$ and its potential energiality $u(x,y,z,t)$ change so randomly and smoothly transform into each other that their sum (i.e., the total mechanical energiality ε) always remains constant.

Condition (45) suggests that in this case, the ChWP never loses its total mechanical energiality for friction with the environment. The reason for such a chaotic behavior of a particle can be random fluctuations of its potential energiality.

Averaging Ex. (45) leads to the condition

$$\varepsilon = \overline{t(x,y,z,t)} + \overline{u(x,y,z,t)} = const\,, \tag{46}$$

and equation (39) takes the form

$$-\frac{\eta_r^2}{2}\nabla^2\psi(\vec r) + \overline{u(\vec r)}\,\psi(\vec r) = \varepsilon\psi(\vec r)\,, \tag{47}$$

In the nonrelativistic massless stochastic quantum mechanics (MSQM), developed in this article, equation (47) is an analogue of the stationary (i.e., time - independent) Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec r) + \overline{U(\vec r)}\,\psi(\vec r) = E\,\psi(\vec r). \tag{48}$$

Let's divide both sides of Eq. (48) by the particle mass *m*

$$-\frac{\hbar^2}{2m^2}\nabla^2\psi(\vec r) + \frac{\overline{U(\vec r)}}{m}\,\psi(\vec r) = \frac{E}{m}\,\psi(\vec r), \tag{49}$$

and take into account definitions (19) – (20). As a result, we represent the stationary Schrödinger equation in the following form

$$-\frac{1}{2}\left(\frac{\hbar}{m}\right)^2\nabla^2\psi(\vec r) + \overline{u(\vec r)}\,\psi(\vec r) = \varepsilon\,\psi(\vec r), \tag{50}$$

Comparing equations (47) and (50), we find that they completely coincide for

$$\eta_r = \frac{2\sigma_r^2}{\tau_{r\,cor}} = \frac{\hbar}{m}. \tag{51}$$

this relationship has already been obtained in Appendix 1, see Ex. (A1.49).

Therefore, we will call Ex. (47) the <u>*massless stationary stochastic Schrödinger-Euler-Poisson equation*</u> for finding the extremal $\psi(x,y,z)$ of the functional

$$\overline{s_r} = \int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\eta_r^2}{2}\psi(x,y,z)\nabla^2\psi(x,y,z) + \psi^2(x,y,z)[\varepsilon - \overline{u(x,y,z)}]\right)dxdydzdt. \tag{52}$$

with energiality balance (46).

2.5 Stochastic quantum operators

Let's show how operators are obtained in massless stochastic quantum mechanics (MSQM). To do this, let us return to considering the model of a chaotically wandering particle (ChWP) shown in Figure 1.

During the chaotic movement of a particle in the vicinity of the conditional center, it constantly changes the direction of its movement. Therefore, a particle at each moment of time has an angular momentum

$$\vec{L} = \vec{r} \times \vec{p}, \tag{53}$$

where $\vec{r}$ is the radius vector from the conditional center to the particle (Figure 1); $\vec{p} = m\vec{v}$ is the instantaneous value and direction of the particle momentum vector.

Let's divide both sides of the vector Ex. (53) by the value *m*, as a result we obtain the angular velocity vector

$$\vec{\omega} = \frac{\vec{L}}{m|\vec{r}|^2} = \frac{\vec{r} \times \vec{v}}{|\vec{r}|^2}. \tag{54}$$

We represent the vector equation (54) in the component form

$$\omega_x = \frac{1}{|\vec{r}|^2}\left(yv_z - zv_y\right), \quad \omega_y = \frac{1}{|\vec{r}|^2}\left(zv_x - xv_z\right), \quad \omega_z = \frac{1}{|\vec{r}|^2}\left(xv_y - yv_x\right), \tag{55}$$

Let's average these components

$$\overline{\omega_x} = \frac{1}{|\vec{r}|^2}\left(y\overline{v_z} - z\overline{v_y}\right), \quad \omega_y = \frac{1}{|\vec{r}|^2}\left(z\overline{v_x} - x\overline{v_z}\right), \quad \omega_z = \frac{1}{|\vec{r}|^2}\left(x\overline{v_y} - y\overline{v_x}\right), \tag{56}$$

We use the coordinate representation of the averaged components of the velocity vector (A2.2) for $n = 1$

$$\overline{v_x} = \int_{-\infty}^{+\infty} \psi(x)\left(-i\eta_x \frac{\partial}{\partial x}\right)\psi(x)dx = \left(-i\eta_x \frac{\partial}{\partial x}\right)\int_{-\infty}^{+\infty} \psi(x)\psi(x)dx, \tag{57}$$

$$\overline{v_y} = \int_{-\infty}^{+\infty} \psi(y)\left(-i\eta_y \frac{\partial}{\partial y}\right)\psi(y)dy = \left(-i\eta_y \frac{\partial}{\partial y}\right)\int_{-\infty}^{+\infty} \psi(y)\psi(y)dy, \tag{58}$$

$$\overline{v_z} = \int_{-\infty}^{+\infty} \psi(z)\left(-i\eta_z \frac{\partial}{\partial z}\right)\psi(z)dz = \left(-i\eta_z \frac{\partial}{\partial z}\right)\int_{-\infty}^{+\infty} \psi(z)\psi(z)dz. \tag{59}$$

and take into account that, for example, in (57)

$$\int_{-\infty}^{\infty} \psi(x)\psi(x)dx = \int_{-\infty}^{\infty} \rho(x)dx = 1. \tag{60}$$

Therefore, identities (57) – (59) are equivalent to massless stochastic operators of the components of the velocity vector

$$\hat{v}_x = \frac{\eta_r}{i}\frac{\partial}{\partial x}, \quad \hat{v}_y = \frac{\eta_r}{i}\frac{\partial}{\partial y}, \quad \hat{v}_z = \frac{\eta_r}{i}\frac{\partial}{\partial z}, \tag{61}$$

here it is taken into account that for the isotropic case $\eta_x = \eta_y = \eta_z = \eta_r$.

Massless stochastic operators (61), taking into account (51), correspond to the operators of the components of the momentum vector of the QM [22]

$$\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}, \quad \hat{p}_y = \frac{\hbar}{i}\frac{\partial}{\partial y}, \quad \hat{p}_z = \frac{\hbar}{i}\frac{\partial}{\partial z}.$$

Substituting (57) – (59) into (56), taking into account (60), we obtain massless stochastic operators of the components of the ChWP angular velocity vector

$$\hat{\omega}_x = \frac{\eta_r}{i|\vec{r}|^2}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \quad \hat{\omega}_y = \frac{\eta_r}{i|\vec{r}|^2}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \quad \hat{\omega}_z = \frac{\eta_r}{i|\vec{r}|^2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \tag{62}$$

which correspond to the quantum mechanical operators of the components of the angular momentum vector [22]

$$\hat{L}_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \quad \hat{L}_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \quad \hat{L}_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

In a spherical coordinate system, stochastic operators (62) have the form

$$\hat{\omega}_x = \frac{\eta_r}{i\,|\vec{r}\,|^2}\left(\sin\varphi\frac{\partial}{\partial\theta} - ctg\theta\,\cos\varphi\frac{\partial}{\partial\varphi}\right),$$
$$\hat{\omega}_y = \frac{\eta_r}{i\,|\vec{r}\,|^2}\left(\cos\varphi\frac{\partial}{\partial\theta} - ctg\theta\sin\varphi\frac{\partial}{\partial\varphi}\right), \tag{63}$$
$$\hat{\omega}_z = \frac{\eta_r}{i\,|\vec{r}\,|^2}\frac{\partial}{\partial\varphi}.$$

The stochastic massless operator of the square of the modulus of the angular velocity of the ChWP is

$$\hat{\omega}^2 = \hat{\omega}_x^2 + \hat{\omega}_y^2 + \hat{\omega}_z^2 = -\frac{\eta_r^2}{|\vec{r}\,|^4}\nabla^2_{\theta,\varphi},$$

where
$$\nabla^2_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{64}$$

All massless stochastic quantum operators massless stochastic quantum mechanics (MSQM), analogous to the QM operators, can be obtained in a similar way. Only in the MSQM, instead of the ratio $\hbar/m$, there is a scale parameter (41), therefore, the BSCM is suitable for describing stochastic processes of any scale.

In a similar way, the mathematical apparatus of the entire massless stochastic quantum mechanics (MSQM) can be built, which almost completely coincides with the mathematical apparatus of the QM. But MSQM is based on the principles of "ordinary" (classical) logic, and is suitable for describing quantum systems and effects of any scale.

2.6 The uncertainty principle in MSQM

The uncertainty in the velocity of a chaotically wandering particle (ChWP) is determined by the standard deviation

$$\sqrt{\overline{v_x^2}} = \sqrt{\int_{-\infty}^{+\infty}\psi(x)\left(-i\eta_x\frac{\partial}{\partial x}\right)^2\psi(x)dx}, \tag{65}$$

and the uncertainty in the particle coordinate is determined by the volatility

$$\sqrt{\overline{x^2}} = \sqrt{\int_{-\infty}^{+\infty} \psi(x) x^2 \psi(x) dx} \ . \tag{66}$$

The joint uncertainty in coordinate and momentum can be represented as

$$\sqrt{\overline{v_x^2 x^2}} = \sqrt{\int_{-\infty}^{+\infty} \psi(x) \left(-i\eta_x \frac{\partial}{\partial x}\right)^2 x^2 \psi(x) dx} = \sqrt{\int_{-\infty}^{+\infty} \psi(x) \left(\eta_x^2 \frac{\partial^2 x^2}{\partial x^2}\right) \psi(x) dx} = \sqrt{2}\eta_x \ . \tag{67}$$

This uncertainty principle of the MSQM is equivalent to the Heisenberg's uncertainty principle $\Delta x \Delta p_x \geq 2\pi\hbar$.

2.7 The time-dependent Schrödinger-Euler-Poisson equation

Let the averaged characteristics of the random trajectory of ChWP change over time, but so slowly that in each small time interval Δt, all these characteristics can be considered unchanged. This unstable behavior of a particle is considered in Appendix 1 and is called a pseudo-stationary random process (PSRP).

Let's also assume that for such a pseudo-stationary stochastic system, the total mechanical energiality (TME) changes insignificantly over time

$$\overline{\varepsilon(x,y,z,t)} = \overline{t(x,y,z,t)} + \overline{u(x,y,z,t)}. \tag{68}$$

In the model under consideration, an insignificant averaged change in the TME of a wandering particle $\pm \overline{d\varepsilon_t(x,y,z,\ t_0 + \Delta t)}$ is associated with a slow change in its kinetic energiality $\pm \overline{dt(x,y,z,\ t_0 + \Delta t)}$ due to external influence in the form of "heating" or "cooling" of the stochastic system.

Since the change in the TME is slow, you can write

$$\overline{\varepsilon(x,y,z,t)} = \overline{\varepsilon(x,y,z,t_0)} \pm \overline{d\varepsilon_t(x,y,z,t_0 + \Delta t)}, \tag{69}$$

the signs (+) or (–) in (69) are associated, respectively, with an increase or a decrease in the averaged TME over time t.

In the future, to reduce the calculations, we will consider the one-dimensional case, without prejudice to the generality of conclusions for the case of three dimensions, and represent (69) in an abbreviated form

$$\overline{\varepsilon(x,t)} = \overline{\varepsilon(x,t_0)} \pm \overline{d\varepsilon_t(x,t_0+\Delta t)}, \qquad (70)$$

where $\overline{d\varepsilon_t(t_0+\Delta t)}$ is the average small change in the TME of a chaotically wandering particle (ChWP), associated with an increase (or decrease) in its average kinetic generality over a short time interval Δ*t*.

In this case, the average efficiency of the ChWP (i.e., the result of dividing both parts of Ex. (6) by the mass of the particle *m*) has the form

$$\overline{s_x}(t) = \int_{t_1}^{t_2} [\overline{t(v_x,t)} - \overline{u(x,t)}dt + \overline{\varepsilon(x,t)}\Delta t, \qquad (71)$$

or, taking into account (70)

$$\overline{s_x}(t) = \int_{t_1}^{t_2} [\overline{t(v_x,t)} - \overline{u(x,t)}dt + [\overline{\varepsilon(x,t_0)} \pm \overline{d\varepsilon_t(x,t)}]\Delta t. \qquad (72)$$

By analogy with (10), we present the average efficiency of (72) as

$$\overline{s_x}(t) = \int_{t_1}^{t_2} \left\{ \frac{1}{2}\int_{-\infty}^{\infty} \rho(v_x,t)v_x^2 dv_x - \int_{-\infty}^{\infty} \rho(x,t)\overline{u(x,t)}dx + \int_{-\infty}^{\infty} \rho(x,t)[\overline{\varepsilon(x,t_0)} \pm \overline{d\varepsilon_t(x,t)}]dx \right\} dt. \qquad (73)$$

Let's write this expression in coordinate representation. For this, we express the PDF $\rho(v_x,t)$ and $\rho(x,t)$ in terms of the probability amplitude $\psi(x,t)$. According to (A1.55) and (A2.2) (see Appendix 1 and Appendix 2) we have:

$$\rho(x,t) = \psi(x,t)\,\psi(x,t), \qquad (74)$$

$$\frac{1}{2}\int_{-\infty}^{\infty} \rho(v_x,t)v_x^2 dv_x \overline{t(x,t)} = \frac{\overline{T(x,t)}}{m} = \overline{t(x,t)} = \frac{1}{2}\overline{v_x^2(x,t)} = \frac{\eta_x^2}{2}\int_{-\infty}^{\infty} \psi(x,t)\frac{\partial^2\psi(x,t)}{\partial x^2}dx, \qquad (75)$$

$$\int_{-\infty}^{\infty} \rho(x,t)\overline{u(x,t)}dx = \frac{\overline{U(x,t)}}{m} = \int_{-\infty}^{\infty} \psi(x,t)\overline{u(x,t)}\psi(x,t)dx, \qquad (76)$$

$$\int_{-\infty}^{\infty} \rho(x,t)[\overline{\varepsilon(x,t_0)} \pm \overline{d\varepsilon_t(x,t)}]dx = \int_{-\infty}^{\infty} \psi(x,t)\overline{\varepsilon(x,t_0)}\psi(x,t)dx \pm i\frac{\eta_x^2}{2D}\int_{-\infty}^{+\infty} \psi(x,t)\frac{\partial\psi(x,t)}{\partial t}dx. \qquad (77)$$

In Ex. (77) it is taken into account that according to (A2.23)

$$\overline{d\varepsilon_t(x,t)} \approx -i\frac{\eta_x^2}{2D}\int_{-\infty}^{+\infty}\psi(x,t)\frac{\partial\psi(x,t)}{\partial t}dx, \tag{78}$$

where D is the imaginary part of the complex diffusion coefficient $B=iD$ of a chaotically wandering particle (ChWP).

Substituting (75) – (77) into (73), we obtain the coordinate representation of the pseudo-stationary averaged efficiency of the ChWP (79)

$$\overline{s_x}(t) = \int_{t_1}^{t_2}\int_{-\infty}^{\infty}\left(\frac{\eta_x^2}{2}\psi(x,t)\frac{\partial^2\psi(x,t)}{\partial x^2} - [\overline{\varepsilon(x,t_0)} - \overline{u(x,t)}]\psi^2(x,t) \pm i\frac{\eta_x^2}{2D}\psi(x,t)\frac{\partial\psi(x,t)}{\partial t}\right)dxdt.$$

Let's find the extremal $\psi(x,t)$ of the functional (79).

First, we recall that the extremality condition of the functional of the form

$$S = \int\int L\left(x,t,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial t},\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial t^2},\frac{\partial^2 z}{\partial t\partial x}\right)dxdt, \quad \text{where} \quad z=\psi(x,t), \tag{80}$$

is determined by the Euler-Poisson equation [21, p. 316]

$$L_z - \frac{\partial}{\partial x}\{L_p\} - \frac{\partial}{\partial t}\{L_g\} + \frac{\partial^2}{\partial x^2}\{L_r\} + \frac{\partial^2}{\partial t^2}\{L_t\} + \frac{\partial^2}{\partial x\partial t}\{L_s\} = 0, \tag{81}$$

where

L_z is the partial derivative of the Lagrangian L with respect to $z=\psi(x,t)$; (82)

L_p is partial derivative of L with respect to $p=\dfrac{\partial z}{\partial x}=\dfrac{\partial\psi(x,t)}{\partial x}$;

L_g is partial derivative of L with respect to $g=\dfrac{\partial z}{\partial t}=\dfrac{\partial\psi(x,t)}{\partial t}$;

L_r is partial derivative of L with respect to $r=\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial^2\psi(x,t)}{\partial x^2}$;

L_t is partial derivative of L with respect to $t=\dfrac{\partial^2 z}{\partial t^2}=\dfrac{\partial^2\psi(x,t)}{\partial t^2}$;

L_s is partial derivative of L with respect to $s=\dfrac{\partial^2 z}{\partial t\partial x}=\dfrac{\partial^2\psi(x,t)}{\partial t\partial x}$,

wherein

$$\frac{\partial}{\partial x}\{L_p\} = L_{px} + L_{pz}\frac{\partial z}{\partial x} + L_{pp}\frac{\partial p}{\partial x} + L_{pg}\frac{\partial g}{\partial x} \quad (83)$$

is full derivative with respect to *x*;

$$\frac{\partial}{\partial t}\{L_g\} = L_{gt} + L_{gz}\frac{\partial z}{\partial t} + L_{gp}\frac{\partial p}{\partial t} + L_{gg}\frac{\partial g}{\partial t}$$

is full derivative with respect to *t*;

$$\frac{\partial^2}{\partial x^2}\{L_r\} = \frac{\partial^2 L_r}{\partial x^2} + L_{rz}\frac{\partial z}{\partial x^2} + L_{rp}\frac{\partial p}{\partial x^2} + L_{rg}\frac{\partial g}{\partial x^2}$$

is full second partial derivative with respect to x^2;

$$\frac{\partial^2}{\partial t^2}\{L_t\} = \frac{\partial^2 L_t}{\partial t^2} + L_{tz}\frac{\partial z}{\partial t^2} + L_{tp}\frac{\partial p}{\partial t^2} + L_{tg}\frac{\partial g}{\partial t^2}$$

is full second partial derivative with respect to t^2;

$$\frac{\partial^2}{\partial t \partial x}\{L_s\} = \frac{\partial^2 L_s}{\partial t \partial x} + L_{sz}\frac{\partial z}{\partial t \partial x} + L_{sp}\frac{\partial p}{\partial t \partial x} + L_{sg}\frac{\partial g}{\partial t \partial x}$$

is full mixed partial derivative with respect to *t* and *x*.

As the Lagrangian *L*, we use the integrand from the time-dependent (pseudo-stationary) averaged efficiency of the ChWP (79)

$$L = \frac{\eta_x^2}{2}\psi(x,t)\frac{\partial^2\psi(x,t)}{\partial x^2} - [\overline{\varepsilon(x,t_0)} - \overline{u(x)}]\psi^2(x,t) \pm i\frac{\eta_x^2}{2D}\psi(x,t)\frac{\partial\psi(x,t)}{\partial t} \quad (84)$$

As a result of substitution of Lagrangian (84) into Ex.s (82) and (83), we obtain

$$L_z = \frac{\eta_x^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} - 2\psi(x,t)[\overline{\varepsilon(x,t_0)} - \overline{u(x,t)}] \pm i\frac{\eta_x^2}{2D}\frac{\partial\psi(x,t)}{\partial t}; \qquad \frac{\partial^2}{\partial t^2}\{L_t\} = 0;$$

$$\frac{\partial}{\partial x}\{L_p\} = 0; \qquad \frac{\partial^2}{\partial x \partial t}\{L_s\} = 0;$$

$$\frac{\partial}{\partial t}\{L_g\} = \pm\, i2\frac{\eta_x^2}{2D}\psi(x,t)\frac{\partial\psi(x,t)}{\partial t}; \qquad \frac{\partial^2}{\partial x^2}\{L_r\} = 2\frac{\eta_x^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}.$$

(85)

Substituting Ex.s (85) into the Euler-Poisson equation (81), we obtain the required equation for determining the extremal $\psi(x,t)$ of the functional (79)

$$\pm\, i\frac{\eta_x^2}{2D}\frac{\partial\psi(x,t)}{\partial t} = \frac{3\eta_x^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} + 2[\overline{\varepsilon(x,t_0)} - \overline{u(x,t)}]\psi(x,t) \quad (86)$$

Generalization to three dimensions, leads to an increase in the number of integrations, while instead of Eq. (86), we get

$$\pm i\frac{\eta_r^2}{2D}\frac{\partial\psi(x,y,z,t)}{\partial t}=\frac{3\eta_r^2}{2}\left\{\frac{\partial^2\psi(x,y,z,t)}{\partial x^2}+\frac{\partial^2\psi(x,y,z,t)}{\partial y^2}+\frac{\partial^2\psi(x,y,z,t)}{\partial z^2}\right\}+ \tag{87}$$
$$+2[\overline{\varepsilon(x,y,z,t_0)}-\overline{u(x,y,z,t)}]\,\psi(x,y,z,t),$$

or in a compact form

$$\pm i\frac{\eta_r^2}{2D}\frac{\partial\psi(\vec r,t)}{\partial t}=\frac{3\eta_r^2}{2}\nabla^2\psi(\vec r,t)+2[\overline{\varepsilon(\vec r,t_0)}-\overline{u(\vec r,t)}]\psi(\vec r,t), \tag{88}$$

where $\vec r$ is the radius vector with the origin in the conditional center of the object under study (Figure 1), ($r^2 = x^2 + y^2 + z^2$); η_r – scale parameter (41) [or (A2.24)];

$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is Laplace operator.

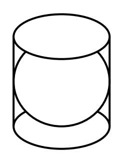
Eq. (88) can be represented as

$$\pm i\frac{\eta_{r1}^2}{3D}\frac{\partial\psi(\vec r,t)}{\partial t}=\eta_{r1}^2\nabla^2\psi(\vec r,t)+2[\overline{\varepsilon(\vec r,t_0)}-\overline{u(\vec r,t)}]\psi(\vec r,t), \tag{89}$$

where according to (41) [or (A2.24)]

$$\eta_{r1}=\sqrt{\frac{3}{2}}\eta_r=\sqrt{\frac{3}{2}}\frac{2\sigma_r^2}{\tau_{r\,cor}}=\sqrt{6}\frac{\sigma_r^2}{\tau_{r\,cor}}. \tag{89a}$$

The ratio of the volume of a cylinder to the volume of a sphere inscribed in it is ***3/2****. Archimedes was so shocked by this discovery that he requested his kinsmen to engrave a sphere inscribed in a cylinder on his tombstone. It is believed that later Cicero found the grave of Archimedes thanks to this symbol (note by S. Petukhov).*



Ex. (89) will be called the <u>massless pseudo-stationary stochastic Euler-Poisson equation</u>.

This equation allows us to find the extremal $\psi(x,y,z,t) = \psi(\vec r,t)$ of the functional of the averaged efficiency of the ChWP

$$\overline{s_r}=\int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\eta_r^2}{2}\psi(\vec r,t)\nabla^2\psi(\vec r)-[\overline{\varepsilon(\vec r,t_0)}-\overline{u(\vec r,t)}]\psi^2(\vec r,t)\pm i\frac{\eta_r^2}{2D}\psi(\vec r,t)\frac{\partial\psi(\vec r,t)}{\partial t}\right)dxdydzdt.$$

with the energiality balance of the investigated pseudo-stationary stochastic system (68), which changes so slowly that in each small time interval $\overline{\varepsilon(x,y,z,t)}$ it can be considered constant.

2.8 Pseudo-stationary (time-dependent) stochastic Schrödinger-Euler-Poisson equation

Both parts of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t)+U(\vec{r},t)\,\psi(\vec{r},t)\,, \tag{90}$$

divided by the particle mass m and multiplied by – 2

$$-i\frac{2\hbar}{m}\frac{\partial\psi(\vec{r},t)}{\partial t}=\frac{\hbar^2}{m^2}\nabla^2\psi(\vec{r},t)-2\frac{\overline{U(\vec{r},t)}}{m}\psi(\vec{r},t).$$

Taking into account (25) and (51), this expression takes the form

$$-i2\eta_r\frac{\partial\psi(\vec{r},t)}{\partial t}=\eta_r^2\nabla^2\psi(\vec{r},t)-2\,\overline{u(\vec{r},t)}\,\psi(\vec{r},t). \tag{90a}$$

On the other hand, if we assume that in Eq. (89), the total mechanical energiality of the ChWP at the initial time moment t_0 is equal to zero, i.e.

$$\overline{\varepsilon(\vec{r},t_0)}=0 \quad \text{and} \quad D=\frac{1}{3}\eta_{r1}=\frac{1}{3}\sqrt{\frac{3}{2}}\eta_r=\sqrt{\frac{2}{3}\frac{\sigma_r^2}{\tau_{r\,cor}}}\,, \tag{91}$$

then this equation takes the form

$$\pm i2\eta_{r1}\frac{\partial\psi(\vec{r},t)}{\partial t}=\eta_{r1}^2\nabla^2\psi(\vec{r},t)-2\,\overline{u(\vec{r},t)}\,\psi(\vec{r},t). \tag{92}$$

Obviously, Eq.s (90a) and (92) differ only in signs in front of their left side and in the value of the scale parameter η_{r1}. Therefore, Eq. (92) will be called the *massless time-dependent stochastic Schrödinger-Euler-Poisson equation* with an imaginary diffusion coefficient $B = iD = i\eta_{r1}/3$.

Recall that the + or – sign on the left side of Eq. (92) depends on the increase or decrease with time of the averaged total mechanical energiality of the considered ChWP [see Ex. (69)].

It is interesting to note that Erwin Schrödinger wrote the equation (4 ") in "Quantisierung als Eigenwertproblem, Vierte Mitteilung", Annalen der Physik (1926) [1] in the following form

$$\Delta\,\psi - \frac{8\pi^2}{h^2}V\,\psi \pm \frac{4\,\pi\,i}{h}\frac{\partial\psi}{\partial t} = 0,$$

Let's rearrange the terms in this expression and take into account that ħ = h/2π,

$$\pm\,2i\hbar\frac{\partial\psi}{\partial t} = \hbar^2\nabla^2\,\psi - 2V\,\psi. \qquad (93)$$

There is a complete analogy (or complete coincidence for ħ = h/2π) between the basic KM Eq. (93) and the MSQM Eq. (92).

3 CONCLUSIONS

The article considers the averaged states of a particle (i.e., a compact body) of any size, which, under the influence of fluctuations in the environment and/or various long-range forces, continuously wanders (oscillates, displaces) in 3 - dimensional space like a Brownian particle.

A constantly trembling (shifting, oscillating) body is represented as a chaotically wandering particle (ChWP) with a continuous trajectory of motion and volume. At the same time, the internal structure of the ChWP is not considered and the deformations of its shape are not taken into account.

These ChWP include the **centers of mass of**: a valence electron in a hydrogen-like atom, a vibrating atom in the crystal lattice, the fluttering heart in the chest of an animal, the trembling yolk in a chicken egg, a floating moth in the vicinity of a burning lamp, swimming fish in the aquarium, a moving mosquito swarm, a flying mosquito in a swarm, a quivering organelle in a biological cell, the oscillating biological cell itself in living tissue, the vibrating iron core in the bowels of the planet, wandering pollen in diluted sugar syrup; an air bubble tossing about in a boiling liquid, a wiggling embryo in the womb; a shifting school of fish in the ocean, a moving astronaut in a space station module, a rushing galaxy in outer space, a fluttering flower in the wind, etc.

All these stochastic systems are similar to each other and obey the same laws, taking into account different types of friction coefficient and viscosity of the medium surrounding the ChWP, as well as the duration of the average period of its behavior. For example, in order to average the chaotic flights of a bird in a cage, a week of continuous observation is required; while averaging the chaotic displacements of the galactic nucleus relative to the main line of its motion in outer space will require millions of years of research. But the results of such observations may turn out to be similar, despite the large difference in the scale of these events.

For example, in §3.6 [27, arXiv:1702.01880], it is predicted theoretically that the possible averaged states of the vibrating nucleus of a biological cell are similar to discrete states of a 3-dimensional quantum mechanical oscillator (i.e., an elementary particle under similar conditions). If these microscopic quantum effects are confirmed experimentally, then we will be able to outline ways to solve the measurement problem in stochastic quantum mechanics.

Within the framework of the MSQM, the problem of studying "pure" states of pico-particles is proposed to be solved as follows. It is necessary to find (or simulate) a stochastic macroscopic system, similar to the investigated picoscopic system (i.e., a chaotically wandering pico-particle), and carry out experiments with the macroscopic system without exerting a tangible effect on it. Then, the results of measurements at the macro level are projected onto possible similar manifestations of the picoscopic system.

Within the MSQM, such an approach to the study of "pure" states of a picoscopic and megascopic systems is possible, since the philosophical foundations of this stochastic mechanics are rooted in antiquity and are based on the belief that all levels of the Universe are similar to each other. In this sense, MSQM is a universal theory for all levels of organization of chaotically oscillating (shifting, trembling, wandering, moving) matter.

As applied to pico-particles (i.e., particles of atomic and subatomic scale), the MSQM corresponds to the stochastic quantum mechanics (SQM) of Edward Nelson [2]. In this case, the MSQM Eq.s (47) and (89), derived in this article on the basis of the principle of the extremum of the "efficiency" of the ChWP, coincided with the corresponding Schrödinger Eq.s (48) and (90) up to coefficients.

In other words, in the massless stochastic quantum mechanics (MSQM), the “pure” wave function $\psi(x,t)$ is the extremal of the functional of the averaged “efficiency” of the ChWP, written in the coordinate representation.

Stochastic Eq.s (47) and (89) have a number of the following advantages over the corresponding Schrödinger equations (48) and (90):

1]. In the reasoning given to derive stochastic Eq.s (47) and (89), no restrictions were imposed on a chaotically wandering particle (ChWP), except for the total energiality balances (46) and (68). That is, ChWP is an ordinary particle that has: volume, trajectory of movement, location and momentum at every moment of time. In other words, the derivation of the stochastic Schrödinger-Euler-Poisson Eq.s (47) and (89) was obtained on the basis of “ordinary” (classical) logic using the theory of probability, the theory of generalized functions, and the calculus of variations (more precisely, the Lagrangian formalism).

Whereas in 95 years, since the appearance of Schrödinger's equations in 1926, many researchers have proposed various methods of deriving them, relying on the axioms of many different interpretations of quantum mechanics, but no universally recognized result has been obtained.

The scientific community has not succeeded in developing logically consistent justifications for the QM axioms. One of the reasons for the general dissatisfaction was the lack of a "beautiful" derivation of the Schrödinger equations.

2]. The reduced Planck's constant ($\hbar = 1.055\cdot 10^{-34}$ J/Hz) limits the scope of the Schrödinger equations (48) and (90), and the entire QM as a whole, to the description of atomic and subatomic scale phenomena.

The fact is that the ratio $\hbar/m$, which is explicitly or latently present in the Schrödinger equations, only then turns out to be physically significant when the particle mass m is very small (for example, it is believed that the electron rest mass $m_e = 9.109\cdot 10^{-31}$ kg).

Whereas the field of application of the stochastic Schrödinger-Euler-Poisson equations (47) and (89) is not limited by anything.

To use Eq.s (47) and (89) to describe the averaged states of any of the above stochastic systems, it is necessary to estimate their scale parameter η_r (41). For this, it is necessary to determine the standard deviation σ_r and the autocorrelation interval $\tau_{r\ cor}$ of a three-dimensional random process, in which the corresponding particle participates, on the basis of sufficiently long observations of the center of mass of the ChWP.

As an example, Appendix 3 shows the possibility of using the massless stationary stochastic Schrödinger-Euler-Poisson equation (47) to obtain quantum numbers characterizing the possible averaged states of a chaotically oscillating nucleus of a biological cell during the interphase period.

3]. The stochastic equation (47) is also applicable to describe the averaged states of a chaotically moving center of mass of an electron in the vicinity of the nucleus of a hydrogen-like atom. If, as a result of statistical processing of indirect observations of the chaotic behavior of a valence electron in such an atom, it turns out that its scale parameter is

$$\eta_{er} = \frac{2\sigma_{er}^2}{\tau_{er\,cor}} \approx \frac{\hbar}{m_e} = \frac{1{,}055\cdot 10^{-34}}{9{,}1\cdot 10^{-31}} \approx 0{,}116\cdot 10^{-3} m^2/s,$$

then Eq.s (47) and (50) for this case will turn out to be almost completely equivalent. In this sense, the time-independent Schrödinger equation (50) can be regarded as a particular case of the stationary stochastic Schrödinger - Euler - Poisson equation (47).

4]. In Schrödinger's equations (48) and (90), the mass of an elementary particle is present. But this mass cannot be directly measured by macroscopic measuring instruments.

On the other hand, in the stochastic Schrödinger-Euler-Poisson equations (47) and (89) there is no particle mass. In this case, the standard deviation σ_r and the autocorrelation interval $\tau_{r\,cor}$ of a three-dimensional random process, in which the ChWP is involved, can always be estimated based on the statistical processing of the results of sufficiently long observations of practically any stochastic system. Therefore, the stochastic Eq.s (39), (47), (89), and (92) obtained in this article are of a universal nature.

MSQM predicts that many stationary random processes (in which ChWP are involved) have the possibility of transition from one stationary state to another with the absorption or release of a certain part of the total mechanical energiality.

This is easy to check, for example, in the case of a moth constantly chaotically flying around a luminous lamp. With a video camera, you can record his chaotic movements for a long time. If you then scroll through the video recording at high speed, then the moth will not be visible on the screen, but there will be a stable blurry dark spot, which reflects the PDF of the location of its center of mass. It should be expected that if the moth is not disturbed by anything, then the blurred spot will resemble a Gaussian PDF with the greatest darkening in the area of the center of the light bulb. However, if the moth is somehow energetically influenced, for example, by heat or ultrasound with a certain frequency, then its average behavior can abruptly change. In this case, the blurred spot can change the configuration to the average shape of a ring or figure-eight, etc.

Also, the center of mass of a flower, depending on the intensity of gusts of wind, can, on average, describe a straight segment, a circle, an ellipse, a figure-eight, or another Lissajous figure.

Similar 2-D and 3-D quantum effects appear in all ChWP of any scale. This contains the main idea of massless stochastic quantum mechanics (MSQM):

“Studying stochastic objects of the macrocosm using conventional (benchtop) methods, we simultaneously obtain information about all similar objects of the microcosm and objects of cosmic scale.

The approach proposed in this paper makes it possible to derive the equations of nonrelativistic massless stochastic quantum mechanics (MSQM) (39), (47), (89), (92) based on principles fundamentally different from the ideological foundations of modern QM interpretations: Copenhagen, Many-worlds, Consistent histories, Decoherence, etc., but the mathematical apparatus of the MSQM turns out to be completely analogous to the mathematical apparatus of the QM.

Apparently, many other equations of quantum field theory can be obtained in a similar way, for example: the Klein-Fock-Gordon equation, the Dirac equation, the Maxwell equations, etc. It is possible that the algorithm for deriving them is similar to the approach given in this work:

1) the deterministic action of the system is recorded;

2) mass is extracted from the action of the system and the "efficiency" of this system is obtained;

3) the efficiency of the system is averaged;

4) all the averaged terms in the integrand of the averaged "efficiency" are represented through the PDF $\rho(x)$;

5) all the terms of the Lagrangian of the averaged "efficiency" of the system are converted into a coordinate representation;

6) the equation for the extremal of the resulting functional is determined by means of the calculus of variations.

It is possible that further research will confirm the validity of this approach to the derivation of the field theory equations.

We hope that this work will assist in the discovery and study of quantum phenomena not only of the microcosm, but also of the macro- and mega Worlds.

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Appendix 1

A1 Determination of the PDF of the derivative of a stationary and pseudo-stationary differentiated random process

Consider several realizations of the random process $\xi(t)$ (Figure A1.1).

In General, this process is non-stationary, but we assume that all the averaged characteristics of this process in the section t_i do not significantly differ from its similar averaged characteristics in the section t_j . That is, we require that all the moments and central moments of this process in the section t_i are approximately equal to the corresponding moments and central moments in the section t_j when $\tau = t_j - t_i$ tending to zero. For example,

$$\overline{\xi(t_i)} \approx \overline{\xi(t_j)}; \qquad (\Pi1.1)$$

$$\overline{\xi^2(t_i)} \approx \overline{\xi^2(t_j)}, \text{ etc.} \qquad (\Pi1.2)$$

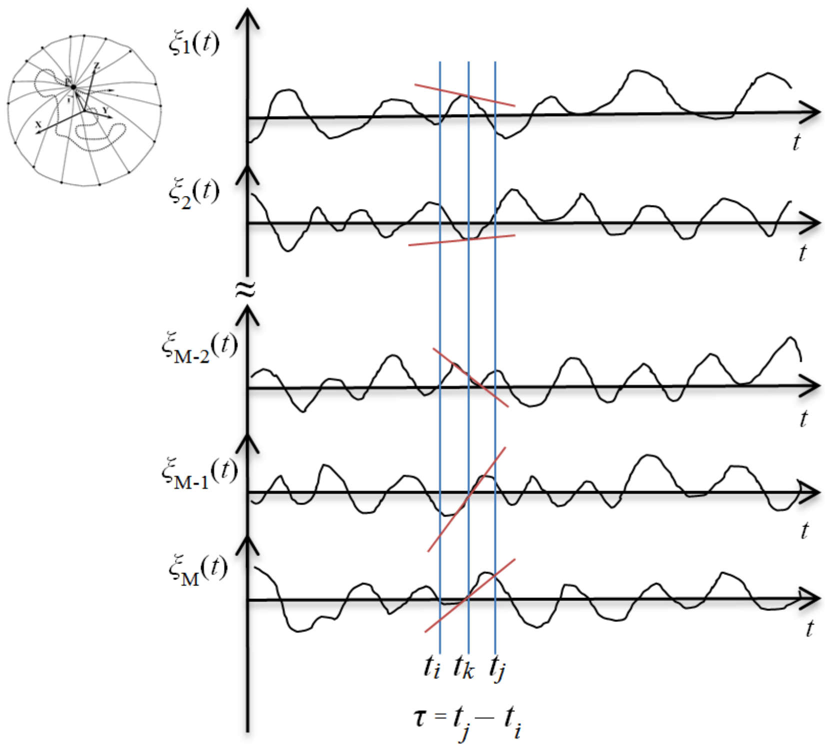

Fig. A1.1. The realizations of a differentiable stationary or pseudo-stationary random process $\xi(t)$. These realizations can be interpreted, for example, as time changes in the projection of the location of a wandering particle on the X axis (see Figure 1), i.e. $x(t) = \xi(t)$

In other words, the considered random process $\xi(t)$ is either stationary or close to it. However, in each section t_m, all the averaged characteristics of such a process remain unchanged. For convenience, we will call such a process "pseudo-stationary random process" (PSRP).

All conclusions about the PSRP, made in this appendix, also apply to the stationary random process (SRP).

There is a known procedure for obtaining the PDF $\rho(\xi'_k)$ of the derivative of a random process $\xi'(t) = d\xi(t)/dt$ with a known two-dimensional PDF of a random stationary process [17, 18]

$$\rho(\xi_i, \xi_j) = \rho(\xi_i, t_i; \xi_j, t_j). \tag{A1.4}$$

In Ex. (A1.4) we make the change of variables

$$\xi_i = \xi_k - \frac{\tau}{2}\xi'_k; \quad \xi_j = \xi_k + \frac{\tau}{2}\xi'_k; \quad t_i = t_k - \frac{\tau}{2}; \quad t_j = t_k + \frac{\tau}{2}, \tag{A1.5}$$

where $\tau = t_j - t_i; \quad t_k = \frac{t_j - t_i}{2}$, with the Jacobian $[J] = \tau$.

As a result, from the two-dimensional PDF (A1.4) we obtain

$$\rho_2(\xi_k, \xi'_k) = \lim_{\tau \to 0} \tau\, \rho_2\left(\xi_k - \frac{\tau}{2}\xi'_k,\ t_k - \frac{\tau}{2};\ \xi_k + \frac{\tau}{2}\xi'_k,\ t_k + \frac{\tau}{2}\right). \tag{A1.6}$$

Integrating (A1.6) over ξ_k, we find the required PDF $\rho(\xi'_k)$ in the section t_k [6]:

$$\rho(\xi'_k) = \int_{-\infty}^{\infty} \rho(\xi_k, \xi'_k)\, d\xi_k. \tag{A1.7}$$

Let's now consider the possibility of obtaining the PDF $\rho(\xi'_k)$ for a known one-dimensional PDF $\rho(\xi)$.

To solve this problem, we use the following properties of random processes:

1. A two-dimensional PDF of a random process can be represented as [17,18]

$$\rho(\xi_i, t_i; \xi_j, t_j) = \rho(\xi_i, t_i)\rho(\xi_j, t_j / \xi_i, t_i), \tag{П1.8}$$

where $\rho(\xi_j, t_j / \xi_i, t_i)$ is the conditional PDF.

2. For any PSRP and SRP the approximate identity is valid

$$\rho(\xi_i, t_i) \approx \rho(\xi_j, t_j). \tag{A1.9}$$

3. The conditional PDF of a random process at $\tau = t_i - t_j$ tending to zero becomes in the delta function

$$\lim_{\tau \to 0} \rho(\xi_j, t_j / \xi_i, t_i) = \delta(\xi_j - \xi_i). \tag{A1.10}$$

Using the above properties, we prepare a random process in the interval $[t_i = t_k - \tau/2;\ t_j = t_k + \tau/2]$ as $\tau \to 0$, based on the following procedure.

The PDF $\rho(\xi_i) = \rho(\xi_i, t_i)$ in the section t_i and the PDF $\rho(\xi_j) = \rho(\xi_j, t_j)$ in the section t_j can always be represented as a product of two functions

$$\rho(\xi_i) = \varphi(\xi_i)\varphi(\xi_i) = \varphi^2(\xi_i), \tag{A1.11}$$

$$\rho(\xi_j) = \varphi(\xi_j)\varphi(\xi_j) = \varphi^2(\xi_j),$$

where $\varphi(\xi_i)$ is the probability amplitude (PA) of the random variable ξ_i in the section t_i; $\varphi(\xi_j)$ is a PA of a random variable ξ_j in the section t_j.

For PSRP, the approximate expression is valid

$$\varphi(\xi_i) \approx \varphi(\xi_j), \qquad \text{(A1.12)}$$

which can be verified by taking the square root of both parts (A1.9).

For SRP, the approximate relation (A1.12) becomes the equality

$$\varphi(\xi_i) = \varphi(\xi_j), \qquad \text{(A1.12a)}$$

Note that the approximate Ex. (A.1.12) at $\tau \to 0$ for the majority of non-stationary random processes (including for PSRP) also turns into the equality

$$\varphi(\xi_i, t_i) = \lim_{\tau\to 0} \varphi(\xi_j,\ t_j = t_i + \tau). \qquad \text{(A1.13)}$$

When the condition (A.1.12) [or (A.1.12a)] is satisfied, Ex. (A1.8) can be represented in the following form

$$\rho(\xi_i, \xi_j) \approx \varphi(\xi_i)\rho(\xi_j / \xi_i)\varphi(\xi_j), \qquad \text{(A1.14)}$$

where $\rho(\xi_j / \xi_i)$ is the conditional PDF.

Let's write (A.1.14) in expanded form

$$\begin{gathered}\rho\left[\xi_i, t_i = t_k - \frac{\tau}{2}; \xi_j, t_j = t_k + \frac{\tau}{2}\right] \approx \\ \approx \left[\xi_i, t_i = t_k - \frac{\tau}{2}\right]\rho\left[\xi_j, t_j = t_k + \frac{\tau}{2} / \xi_i, t_i = t_k - \frac{\tau}{2}\right]\varphi\left[\xi_j, t_j = t_k + \frac{\tau}{2}\right].\end{gathered} \qquad \text{(A1.15)}$$

Let τ tend to zero in (A1.15), so that the given time interval contracts uniformly on the left and right at the middle moment of time $t_k = (t_j - t_i)/2$. In this case, taking into account (A1.10), from (A1.14), we obtain the exact equality

$$\lim_{\tau\to 0} \rho(\xi_i, \xi_j) = \lim_{\tau\to 0}\{\varphi(\xi_i)\rho(\xi_j / \xi_i)\varphi(\xi_j)\} = \varphi(\xi_{ik})\delta(\xi_{jk} - \xi_{ik})\varphi(\xi_{jk}), \qquad \text{(A1.16)}$$

where ξ_{ik} is the result of the tendency of the random variable $\xi(t_i)$ to the random variable $\xi(t_k)$ on the left; ξ_{jk} is the result of the tendency of the random variable $\xi(t_j)$ to the random variable $\xi(t_k)$ from the right.

Integrating both sides of Ex. (A1.16) over ξ_{ik} and ξ_{jk}, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\xi_{ik})\delta(\xi_{jk}-\xi_{ik})\varphi(\xi_{jk})d\xi_{ik}d\xi_{jk}=1. \qquad (\Pi1.17)$$

In (A1.17), the properties of the δ-function are taken into account.

Let's set the specific form of the δ-function. To do this, consider a random Markov process for which the Fokker - Planck equation is valid

$$\frac{\partial\rho(\xi_j/\xi_i)}{\partial t}=B\frac{\partial^2\rho(\xi_j/\xi_i)}{\partial\xi^2}, \qquad (\Pi1.18)$$

where B is the diffusion coefficient.

One of the solutions of this differential equation, as is well known, has the form

$$\rho\left(\xi_j,t_j/\xi_i,t_i\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{iq(\xi_j-\xi_i)-q^2B(t_j-t_i)\}dq, \qquad (\Pi1.19)$$

where q is the generalized frequency.

For $\tau = t_j - t_i \to 0$ from (A.1.19) we obtain one of the definitions of the δ-function

$$\lim_{\tau\to 0}\rho(\xi_j/\xi_i)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{iq(\xi_{jk}-\xi_{ik})\}dq=\delta(\xi_j-\xi_i). \qquad (\Pi1.20)$$

This result was obtained for the case as $\tau \to 0$. Therefore, the δ-function (A1.20) can correspond not only to a Markov random process, but also to many other stationary and non-stationary random processes. In other words, one could immediately assume that the δ-function for the PSRP has the form (A.1.20) without referring to the Fokker - Planck equation (A.1.18).

Let's substitute δ-function (A.1.20) into Ex. (A.1.17)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\xi_{ik})\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{iq(\xi_{jk}-\xi_{ik})\}dq\right]\varphi(\xi_{jk})d\xi_{ik}\,d\xi_{jk}=1, \qquad (A1.21)$$

and change the order of integration in (A1.21)

$$\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\xi_{ik})\exp\{-iq\xi_{ik}\}d\xi_{ik}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\xi_{jk})\exp\{iq\xi_{jk})d\xi_{jk}\right]dq=1. \quad \text{(A1.22)}$$

Let's take into account that, according to (A1.13), for the SRP and PSRP the condition $\varphi(\xi_{ik}) = \varphi(\xi_{jk})$ is fulfilled. Therefore, Ex. (A.1.22) can be represented as

$$\int_{-\infty}^{\infty}\varphi(q)\varphi^*(q)dq=1, \quad \text{(A1.23)}$$

where
$$\varphi(q)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\xi_k)\exp\{-iq\xi_k\}d\xi_k, \quad \text{(A1.24)}$$

$$\varphi^*(q)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\xi_k)\exp\{iq\xi_k\}d\xi_k. \quad \text{(A1.25)}$$

The integrand $\varphi(q)\varphi^*(q)$ in (A1.23) meets all the requirements of the PDF $\rho(q)$ of the random variable q:

$$\rho(q)=\varphi(q)\varphi^*(q)=\left|\varphi(q)\right|^2. \quad \text{(A1.26)}$$

Let's clarify the physical meaning of q.

The features of the considered random process impose the following restrictions on the generalized frequency q:

1) the random variable q should characterize the random process $\xi(t)$ in the section t_k (Figure A1.1), i.e. in the interval $\tau = t_j - t_i$ tending to 0;

3) the variable q must belong to the set of real numbers, that is, take any value from the range $]-\infty, \infty[$.

These requirements are satisfied by the following random values associated with the PSRP (or SRP) in the time interval τ:

$$\xi_i'=\frac{\partial\xi_k}{\partial t},\ \xi_i''=\frac{\partial^2\xi_k}{\partial t},\ \dots,\ \xi_i^{(n)}=\frac{\partial^n\xi_i}{\partial t^n}. \quad \text{(A1.27)}$$

To clarify which of these values is associated with the generalized frequency q, consider one implementation of the process under study (see Figure A1.1). The function $\xi(t)$ in the interval at $\tau < \tau_{cor}$ [where τ_{cor} is the autocorrelation interval of the random process $\xi(t)$] can be expanded in the Maclaurin series

$$\xi(t_j) = \xi(t_i) + \xi'(t_i)\tau + \frac{\xi''(t_i)}{2}\tau^2 + \ldots + \frac{\xi^{(n)}(t_i)}{n!}\tau^n + \ldots \qquad \text{(A1.28)}$$

The Ex. (A1.28) is presented in the following form

$$\frac{\xi_j - \xi_i}{\tau} = \xi_i' + \frac{\xi_i''}{2!}\tau + \ldots + \frac{\xi_i^{(n)}\tau^{n-1}}{n!} + \ldots \qquad \text{(A1.29)}$$

where $\xi(t_i) = \xi_i$, $\xi(t_j) = \xi_j$, and we tend τ to zero.

In this case (A1.29) is reduced to the expression

$$\lim_{\tau\to 0}\frac{\xi_j - \xi_i}{\tau} = \xi_k', \qquad \text{(A1.30)}$$

where $\xi_k = \xi(t_k)$ (see Figure A1.1).

Therefore, it remains to assume that the generalized frequency q in Ex.s (A1.23) – (A1.26) is linearly related only to ξ'_k, i.e.

$$q = \frac{\xi_k'}{\eta}, \qquad \text{(A1.31)}$$

where η is the scale parameter.

The Ex. (A1.31) can be obtained in another way.

Each exponential, for example, from the integral (A1.24), corresponds to a harmonic function with frequency q

$$\exp\{-iq\xi(t)\} \to \xi_k(t) = A\sin(qt), \qquad \text{(A1.32)}$$

this is one of the harmonic components of the random process $\xi(t)$.

Differentiating (A1.32), we obtain $\xi_k'(t) = qA\cos(qt)$, whence it follows

$$q = \lim_{t\to 0}\frac{\xi_k'}{A\cos(qt)} = \frac{\xi_k'}{A}. \qquad \text{(A1.33)}$$

For $A = \eta$, Ex.s (A1.31) and (A1.33) coincide.

Substituting (A1.31) into (A1.23) – (A1.26), we obtain the following required procedure for obtaining the PDF $\rho(\xi',t)$ of a pseudo-stationary random process (PSRP) or stationary random process (SRP) $\xi(t)$ in any section t_k for a known one-dimensional PDF $\rho(\xi,t)$ in the same section:

1. A given one-dimensional PDF $\rho(\xi,t)$ is represented as a product of two probability amplitudes (PA) $\varphi(\xi)$:

$$\rho(\xi,t)=\varphi(\xi,t)\varphi(\xi,t). \tag{A1.34}$$

2. Two Fourier transforms are performed

$$\varphi(\xi',t)=\frac{1}{\sqrt{2\pi\eta}}\int_{-\infty}^{\infty}\varphi(\xi,t)\exp\{i\xi'\xi/\eta\}d\xi, \tag{A1.35}$$

$$\varphi*(\xi',t)=\frac{1}{\sqrt{2\pi\eta}}\int_{-\infty}^{\infty}\varphi(\xi,t)\exp\{-i\xi'\xi/\eta\}d\xi. \tag{A1.36}$$

3. Finally, for an any section of the PSRP (or SRP), we obtain the required PDF of its derivative

$$\rho(\xi',t)=\varphi(\xi',t)\varphi^*(\xi',t)=|\varphi(\xi',t)|^2. \tag{A1.37}$$

Once again, we note that the procedure (A1.34) – (A1.37) can be applied to any stationary and pseudo-stationary random processes {i.e., random processes with a slowly varying PDF $\rho(\xi,t)$}, for which, as $\tau\to 0$, the δ-function takes the form (A1.20).

To clarify the physical meaning of the scale parameter η, consider a stationary random process $\xi(t)$ with a Gaussian distribution of the random variable ξ

$$\rho(\xi)=\frac{1}{\sqrt{2\pi\sigma_\xi^2}}\exp\{-(\xi-a_\xi)^2/2\sigma_\xi^2\}, \tag{A1.38}$$

where $\sigma_\xi{}^2$ and a_ξ are the variance and mathematical expectation of the process.

Performing the sequence of operations (A1.34) – (A1.37) with the PDF (A1.38), we obtain the PDF of the derivative of this random process

$$\rho(\xi')=\frac{1}{\sqrt{2\pi[\eta/2\sigma_\xi]^2}}\exp\left\{-\frac{\xi'^2}{2[\eta/2\sigma_\xi]^2}\right\}. \tag{A1.39}$$

On the other hand, using the well-known procedure (A1.4) – (A1.7) for a similar case, we obtain [18]

$$\rho(\xi') = \frac{1}{\sqrt{2\pi\sigma_{\xi'}^2}} \exp\left\{-\xi'^2 / 2\sigma_{\xi'}^2\right\}, \tag{A1.40}$$

where $$\sigma_{\xi'} = \sigma_{\xi} / \tau_{cor},$$

here τ_{cor} is the autocorrelation interval of the initial random process $\xi(t)$.

Comparing the PDF (A1.39) and (A1.40), we find that

$$\eta = \frac{2\sigma_{\xi}^2}{\tau_{\xi\, cor}} \tag{A1.41}$$

The Ex. (A1.41) was obtained for a Gaussian random process, but the standard deviation σ_{ξ} and the autocorrelation interval $\tau_{\xi\, cor}$ are the main characteristics of any SRP or PSRP. All other moments and central moments in the case of a non-Gaussian distribution of the random variable $\xi(t)$ will make an insignificant contribution to the Ex. (A1.41). Therefore, it can be argued with a high degree of reliability that Ex. (A1.41) is applicable for a large class of stationary and pseudo-stationary random processes.

In quantum mechanics for the transition from the coordinate representation of the wave function of a pico-particle to its momentum representation, there is the procedure [10]

$$\varphi(p_x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(x) \exp\{ip_x x / \hbar\} dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(x) \exp\{imx'x / \hbar\} dx, \tag{A1.42}$$

$$\varphi * (p_x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(x) \exp\{-ip_x x / \hbar\} dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(x) \exp\{-imx'x / \hbar\} dx, \tag{A1.43}$$

where $\hbar$ = $1.055 \cdot 10^{-34}$ J/Hz is the reduced Planck's constant, and it is also taken into account that the x-component of the particle momentum p_x is related with its speed v_x (i.e., time derivative)

$$p_x = mv_x = m\frac{dx}{dt} = mx', \tag{A1.45}$$

In the case when

$$\eta_x = \frac{2\sigma_x^2}{\tau_{x cor}} = \frac{\hbar}{m} \text{ with dimension } (m^2/s), \qquad \text{(A1.46)}$$

procedures (A1.34) – (A1.37) and (A1.42) – (A1.43) completely coincide.

From Ex. (A1.46) it follows that Planck's constant can be expressed through the main averaged parameters σ_x and $\tau_{x\ cor}$ of a stationary (or pseudo-stationary) random process, which involves a randomly wandering pico-particle (for example, an electron).

At the same time, the field of application of the procedure (A1.42) – (A1.43) is limited by the smallness of the reduced Planck's constant $\hbar$. While the procedure (A1.34) – (A1.37) can be applied for random stationary and pseudo - stationary processes of any scale. Such random processes include chaotic oscillations of the center of mass of the nucleus of a biological cell, chaotic movements of the tip of a tree branch, chaotic change in the position of the center of mass of the planet's nucleus, etc.

Let's note the following intermediate results:

1]. For a stationary and pseudo-stationary random process $\xi(t) = x(t)$, the following procedure for obtaining the PDF $\rho(x')$ derivative of this process can be applied.

A given one-dimensional PDF $\rho(x)$ of a stationary process [or a slowly varying PDF $\rho(x,t)$ of a pseudo-stationary process] is represented as a product of two PA $\varphi(x)$ or $\varphi(x,t)$:

$$\rho(x) = \varphi(x)\varphi(x) \quad \text{or} \quad \rho(x,t) = \varphi(x,t)\varphi(x,t). \qquad \text{(A1.47)}$$

a) For a stationary random process (SRP), two Fourier transforms are performed

$$\varphi(x') = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \varphi(x) \exp\{ix'x/\eta_x\}dx = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \varphi(x) \exp\{iv_x x/\eta_x\}dx, \quad \text{(A1.48)}$$

$$\varphi*(x') = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \varphi(x) \exp\{-ix'x/\eta_x\}dx = \frac{1}{\sqrt{2\pi\eta_x}} \int_{-\infty}^{\infty} \varphi(x) \exp\{-iv_x x/\eta_x\}dx. \quad \text{(A1.49)}$$

and the desired PDF of the derivative of this process is determined

$$\rho(x') = \varphi(x')\varphi^*(x') = |\varphi(x')|^2. \quad \text{(A1.50)}$$

or
$$\rho(v_x) = \varphi(v_x)\varphi^*(v_x) = |\varphi(v_x)|^2 \quad \text{(A1.51)}$$

where
$$\eta_x = \frac{2\sigma_x^2}{\tau_{x\kappa op}} = \frac{\hbar}{m} \quad \text{(A1.52)}$$

σ_x is the standard deviation of the initial stationary random process $x(t)$;

$\tau_{x\,cor}$ is the autocorrelation interval of this process.

In § 2.6 of the article [11, arXiv:2007.13527], the procedure (A1.47) – (A1.51) is applied to obtain the PDF ρ(x′) of the derivative of stationary random processes with distribution laws: Gaussian, uniform, Laplace, Cauchy and sinusoidal.

b) For a pseudo-stationary random process (PSRP), two Fourier transforms are performed (A1.53)

$$\varphi(x',t) = \frac{1}{\sqrt{2\pi\eta_x(t)}} \int_{-\infty}^{\infty} \varphi(x,t) \exp\{ix'x/\eta_x(t)\}dx = \frac{1}{\sqrt{2\pi\eta_x(t)}} \int_{-\infty}^{\infty} \varphi(x,t) \exp\{iv_x x/\eta_x(t)\}dx,$$

$$\varphi*(x',t) = \frac{1}{\sqrt{2\pi\eta_x(t)}} \int_{-\infty}^{\infty} \varphi(x,t) \exp\{-ix'x/\eta_x(t)\}dx = \frac{1}{\sqrt{2\pi\eta_x(t)}} \int_{-\infty}^{\infty} \varphi(x,t) \exp\{-iv_x x/\eta_x(t)\}dx.$$
(A1.54)

and the required PDF of the derivative of this process is determined at each time moment *t*

$$\rho(x',t) = \varphi(x',t)\varphi^*(x',t) = |\varphi(x',t)|^2. \quad \text{(A1.55)}$$

or
$$\rho(v_x,t) = \varphi(v_x,t)\varphi^*(v_x,t) = |\varphi(v_x,t)|^2 \quad \text{(A1.56)}$$

where
$$\eta_x(t) = \frac{2\sigma_x^2(t)}{\tau_{x\,cor}(t)} = \frac{\hbar}{m(t)} \quad \text{(A1.57)}$$

$\sigma_x(t)$ is the standard deviation of the initial pseudo-stationary random process $x(t)$ from its mean value at time t;

$\tau_{x\,cor}(t)$ is the autocorrelation interval of this process at time t.

2]. The procedure (A1.47) – (A1.52) up to the proportionality coefficient η coincides with the quantum-mechanical procedure (A1.42) – (A1.43) of transition from the coordinate representation to the impulse one. But the quantum - mechanical procedure (A1.42) – (A1.43) was obtained using a rather unobvious (exotic) hypothesis about the possible existence of de Broglie's waves of matter (which were never discovered). While the procedure (A1.47) – (A1.52) is obtained on the basis of a detailed analysis of a differentiable random process with the only assumption (which may be questioned) that the δ-function has the form (A1.20). In this regard, it is interesting to analyze which procedures for the transition from PDF $\rho(x)$ to PDF $\rho(x')$ can lead to other types of δ-function?

Also, there is no need to use Louis de Broglie's hypothesis of the existence of matter waves to describe the diffraction of particles by a crystal. In the article [11, arXiv:2007.13527] it is shown that, based on the laws of geometric optics and the theory of probability, a formula was obtained for calculating the volumetric scattering diagrams of particles on a multilayer periodic surface of a crystal.

3]. In the case of studying the chaotic behavior of pico-particles, the ratio $\hbar/m$ can be expressed through the main characteristics of the investigated random process (A1.46). In the author's opinion, this is a very important result, since it is practically impossible to estimate the real mass of a mobile elementary particle. Let's recall that in physical reference books only the rest masses of elementary particles are given, which are determined indirectly on the basis of complex experiments. Whereas it is much easier to obtain an estimate of the standard deviation σ_x and the autocorrelation interval $\tau_{x\,cor}$ of a randomly wandering particle. It is also important that the reduced Planck constant $\hbar$ loses its fundamental character and turns out to be the dimensional coefficient of proportionality between the particle mass and the ratio of the averaged characteristics of the random process.

Appendix 2

A.2 Coordinate representation of the average speed a chaotically wandering particle

For stationary and pseudo-stationary random processes (see Appendix1), we prove the validity of equalities (A2.1)

$$\overline{x'^n} = \overline{v_x^n} = \int_{-\infty}^{+\infty} \rho(x')x'^n dx'_x = \int_{-\infty}^{+\infty} \rho(v_x)v_x^n dv_x = \int_{-\infty}^{+\infty} \psi(v_x)v_x^n \psi^*(v_x)dv_x = \int_{-\infty}^{+\infty} \psi(x)\left(-i\eta_x \frac{\partial}{\partial x}\right)^n \psi(x)dx,$$

$$\overline{x'^n}(t) = \overline{v_x^n}(t) = \int_{-\infty}^{+\infty} \rho(x',t)x'^n dx'_x = \int_{-\infty}^{+\infty} \rho(v_x,t)v_x^n dv_x = \int_{-\infty}^{+\infty} \psi(x,t)\left(-i\eta_x(t) \frac{\partial}{\partial x}\right)^n \psi(x,t)dx, \quad \text{(A2.2)}$$

where n is an integer, positive degree; η_x is the scale parameter (A1.52).

Experts in the field of QM are well aware of the proof of a similar expression

$$\overline{p_x^n} = \int_{-\infty}^{+\infty} \rho(p_x)p_x^n dp_x = \int_{-\infty}^{+\infty} \psi(p_x)p_x^n \psi(p_x)dp_x = \int_{-\infty}^{+\infty} \psi(x)\left(-i\hbar \frac{\partial}{\partial x}\right)^n \psi(x)dx,$$

see, for example, [20]. However, in view of the importance of this proof for this article, we present it in a slightly modified form, as applied to the features of massless stochastic quantum mechanics (MSQM).

Let's use the Fourier transforms (A1.48) and (A1.48)

$$\psi(v_x) == \int_{-\infty}^{+\infty} \psi(x) \frac{e^{i\frac{v_x x}{\eta_x}}}{(2\pi\eta_x)^{1/2}}dx = \int_{-\infty}^{+\infty} \psi(x) \frac{e^{i\frac{p_x x}{\hbar}}}{(2\pi\hbar)^{1/2}}dx, \quad \text{(A2.3)}$$

$$\psi^*(v_x) = \int_{-\infty}^{+\infty} \psi(x) \frac{e^{-i\frac{v_x x}{\eta_x}}}{(2\pi\eta_x)^{1/2}}dx = \int_{-\infty}^{+\infty} \psi(x) \frac{e^{-i\frac{p_x x}{\hbar}}}{(2\pi\hbar)^{1/2}}dx, \quad \text{(A2.4)}$$

Substitute integrals (A2.3) and (A2.4) into the third part of Eq. (A2.1)

$$\overline{v_x^n} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\psi(x_i)\frac{e^{-i\frac{v_x x_i}{\eta_x}}}{(2\pi\eta_x)^{1/2}}dx_i\, v_x^n\int_{-\infty}^{+\infty}\psi(x_j)\frac{e^{i\frac{v_x x_j}{\eta_x}}}{(2\pi\eta_x)^{1/2}}dx_j\right]dp_x. \qquad \text{(A2.5)}$$

It is easy to verify by direct verification that

$$v_x^n e^{i\frac{v_x x_j}{\eta_x}} = \left(-i\eta_x\frac{\partial}{\partial x_j}\right)^n e^{i\frac{v_x x_j}{\eta_x}}, \text{ or } \quad p_x^n e^{i\frac{p_x x_j}{\hbar}} = \left(-i\hbar\frac{\partial}{\partial x_j}\right)^n e^{i\frac{p_x x_j}{\hbar}}. \qquad \text{(A2.6)}$$

Let's rewrite (A2.5) taking into account (A2.6)

$$\overline{v_x^n} = \frac{1}{2\pi\eta_x}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\psi(x_i)e^{-i\frac{v_x x_i}{\eta_x}}dx_i\int_{-\infty}^{+\infty}\psi(x_j)\left(-i\eta_x\frac{\partial}{\partial x_j}\right)^n e^{i\frac{v_x x_j}{\eta_x}}dx_j\right]dv_x. \qquad \text{(A2.7)}$$

We integrate the second integral in the integrand *n* times by parts, while we assume that $\psi(x)$ and its derivatives vanish at the integration boundaries $x = \pm\infty$. Performing these actions, we get

$$\overline{v_x^n} = \frac{1}{2\pi\eta_x}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\psi(x_i)e^{-i\frac{v_x x_j}{\eta_x}}dx_i\int_{-\infty}^{+\infty}e^{i\frac{v_x x_j}{\eta_x}}\left(-i\eta_x\frac{\partial}{\partial x_j}\right)^n\psi(x_j)dx_j\right]dv_x, \quad \text{(A2.8)}$$

or
$$\overline{v_x^n} = \frac{1}{2\pi\eta_x}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}dx_i\int_{-\infty}^{+\infty}dx_j\psi(x_i)e^{i\frac{v_x(x_j-x_i)}{\eta_x}}\left(-i\eta_x\frac{\partial}{\partial x_j}\right)^n\psi(x_j)\right]dv_x. \qquad \text{(A2.9)}$$

Let’s change the order of integration in (A2.9), i.e. first we will integrate over v_x

$$\overline{v_x^n} = \int_{-\infty}^{+\infty}dx_i\int_{-\infty}^{+\infty}dx_j\psi(x_i)\left(-i\hbar\frac{\partial}{\partial x_j}\right)^n\psi(x_j)\frac{1}{2\pi\eta_x}\int_{-\infty}^{+\infty}e^{i\frac{v_x(x_j-x_i)}{\eta_x}}dv_x.$$

There is a delta function in that expression

$$\delta(x_j - x_i) = \frac{1}{2\pi\eta_x}\int_{-\infty}^{+\infty}e^{i\frac{v_x(x_j-x_i)}{\eta_x}}dv_x \text{ type } \delta(x_j - x_i) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{iq(x_j-x_i)}dq. \qquad \text{(A2.10)}$$

Therefore, we represent it in the form

$$\overline{v_x^n} = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} \psi(x_i)\left(-i\eta_x \frac{\partial}{\partial x_j}\right)^n \psi(x_j)\delta(x_j - x_i)dx_j. \tag{A2.11}$$

Using the properties of the δ-function, we finally write

$$\overline{v_x^n} = \int_{-\infty}^{+\infty} \psi(x)\left(-i\eta_x \frac{\partial}{\partial x}\right)^n \psi(x)dx, \tag{A2.12}$$

thus, Ex. (A2.1) is proved for the case of a stationary random process (SSP).

For a pseudo-stationary random process (PSSP), Ex. (A2.2) is proved similarly. Performing operations similar to (A2.5) – (A2.15) using transformations (A1.53) and (A1.54), we obtain

$$\overline{v_x^n}(t) = \int_{-\infty}^{+\infty} \psi(x,t)\left(-i\eta_x(t) \frac{\partial}{\partial x}\right)^n \psi(x,t)dx, \tag{A2.13}$$

Let's return to the consideration of the conditional PDF (A1.19)

$$\rho\left(x_j, t_j / x_i, t_i\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{iq(x_j - x_i) - q^2 B(t_j - t_i)\}dq, \tag{A2.14}$$

where according to (A1.31) $q = \dfrac{x'}{\eta_x} = \dfrac{v_x}{\eta_x}$.

For $\Delta x = x_j - x_i \rightarrow 0$ from (A2.14) we obtain

$$\lim_{\Delta x \to 0} \rho(x_j, t_j / x_i, t_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-\frac{v_x^2}{\eta_x^2} B(t_j - t_i)\}dv_x. \tag{A2.15}$$

Let's take into account that

$$\frac{v_x^2}{2} = t_x = \varepsilon_x, \tag{A2.16}$$

where t_x is the kinetic energiality (21) equal to the total mechanical energiality ε_x (19) in the absence of the potential energiality u_x (20) (i.e., at $u_x = 0$).

For some stochastic processes, it should be assumed that the diffusion coefficient B is a complex number, i.e. $B = iD$. In this case, Ex. (A2.15) taking into account (A2.16) takes the form of the δ_ε-function

$$\lim_{\Delta x\to 0}\rho(x_j,t_j/x_i,t_i)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\varepsilon_x}}\exp\{-i\frac{\varepsilon_x}{\eta_x^2}2D(t_j-t_i)\}d\varepsilon_x=\delta_\varepsilon(t_j-t_i). \quad \text{(A2.17)}$$

Substitute this δ_ε-function into an expression similar to (A.1.17)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x,t_{ik})\delta_\varepsilon(t_j-t_i)\varphi(x,t_{jk})dt_{ik}dt_{jk}=1. \quad \text{(A2.18)}$$

Note that to obtain Ex. (A2.18), it is necessary to consider the evolution of a random process not in time (as shown in Figure A1.1), but in space. This is similar to Feynman's chaotic trajectories of a particle when it moves from one point in space to another [24].

From Ex. (A2.18) by analogy with (A1.22) – (A1.25) follow the Fourier transforms

$$\varphi(\varepsilon_x)=\frac{1}{\sqrt{\pi\frac{\eta_x^2}{D}}}\int_{-\infty}^{\infty}\varphi(x,t)\exp\{i\varepsilon_x t\frac{2D}{\eta_x^2}\}dt, \quad \text{(A2.19)}$$

$$\varphi^*(\varepsilon_x)=\frac{1}{\sqrt{\pi\frac{\eta_x^2}{D}}}\int_{-\infty}^{\infty}\varphi(x,t)\exp\{-i\varepsilon_x t\frac{2D}{\eta_x^2}\}dt. \quad \text{(A2.20)}$$

Considering

$$\varepsilon_x^n e^{i\varepsilon_x t\frac{2D}{\eta_x^2}}=\left(-i\frac{\eta_x^2}{2D}\frac{\partial}{\partial t}\right)^n e^{i\varepsilon_x t\frac{2D}{\eta_x^2}}, \quad \text{(A2.21)}$$

it can be shown that the average change in the kinetic energiality of a particle participating in the PSSP has the form

$$\overline{d\varepsilon_x^n}=\int_{-\infty}^{+\infty}\rho(\varepsilon_x)d\varepsilon_x^n=\int_{-\infty}^{+\infty}\psi(\varepsilon_x)d\varepsilon_x^n\psi^*(\varepsilon_x)dv_x=\int_{-\infty}^{+\infty}\psi(x,t)\left(-i\frac{\eta_x^2}{2D}\frac{\partial}{\partial t}\right)^n\psi(x,t)dx, \quad \text{(A2.22)}$$

and for n = 1, we have

$$\overline{d\varepsilon_x}=-i\frac{\eta_x^2}{2D}\int_{-\infty}^{+\infty}\psi(x,t)\frac{\partial\psi(x,t)}{\partial t}dx. \quad \text{(A2.23)}$$

The proof of the validity of Ex. (A2.22) is similar to the proof of Ex. (A2.2). The similarity of these proofs follows from the symmetry between $p_x x$ and Et (or $v_x x$ and εt) in the de Broglie wave

$$\psi = exp\{-i(p_x x - Et)/\hbar\} = exp\{-i(v_x x - \varepsilon t)m/\hbar\} = exp\{-i(v_x x - \varepsilon t)\eta_x\}. \quad \text{(A2.24)}$$

In the case when the scale parameter $\eta_x(t)$ changes with time, then you can write

$$\overline{d\varepsilon_x} = \int_{-\infty}^{+\infty} \psi(x,t)\left(-\frac{i}{2D}\frac{\partial}{\partial x}\right)\eta_x^2(t)\psi(x,t)dx, \quad \text{(A2.25)}$$

At the same time, a situation is possible when the variance changes with time according to the law $\sigma_x^2(t-t_0)$, and the autocorrelation coefficient changes according to the same law $\tau_{x\,cor}(t-t_0)$. Then

$$\eta_x(t) = \frac{2\sigma_x^2(t)}{\tau_{x\,\kappa op}(t)} \approx \frac{2\sigma_x^2(t-t_0)}{\tau_{x\,\kappa op}(t-t_0)} \approx \eta_x = \frac{2\sigma_x^2}{\tau_{x\,\kappa op}} = const. \quad \text{(A2.26)}$$

In this case (A2.25) again takes the form (A2.23).

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