Appendix 2

The Calculation of Integrals

Calculate the integrals (2.73)

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \int_0^{l_2} \sqrt{\frac{2}{l_2}} \sin(\pi n_1 \xi / l_2) \exp\{i\xi'\xi/\eta\} d\xi \,, \tag{A.2.1}$$

$$\psi^*(\xi') = \frac{1}{\sqrt{2\pi}} \int_0^{l_2} \sqrt{\frac{2}{l_2}} \sin(\pi n_1 \xi / l_2) \exp\{-i\xi'\xi/\eta\} d\xi \tag{A.2.2}$$

We start with the integral (A.2.1), and use the formula $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and represent (A.2.1) in the form

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_0^{l_2} \frac{e^{i\pi n_1 \xi/l_2} - e^{-i\pi n_1 \xi/l_2}}{2i} e^{i\xi'\xi/\eta} d\xi \tag{A.2.3}$$

Let's perform the following transformations

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_0^{l_2} \frac{e^{i\pi n_1 \xi/l_2} e^{i\xi'\xi/\eta} - e^{-i\pi n_1 \xi/l_2} e^{i\xi'\xi/\eta}}{2i} d\xi$$

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_{-\infty}^{\infty} \frac{e^{i\pi n_1 \xi/l_2 + i\xi'\xi/\eta} - e^{-i\pi n_1 \xi/l_2 + i\xi'\xi/\eta}}{2i} d\xi$$

$$\psi(\xi') = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_0^{l_2} \frac{e^{i\xi(\pi n_1/l_2 + \xi'/\eta)} - e^{-i\xi(\pi n_1/l_2 - \xi'/\eta)}}{2i} d\xi$$

$$\psi(\xi') = \frac{1}{2i\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \int_0^{l_2} e^{i\xi(\pi n_1/l_2 + \xi'/\eta)} - e^{-i\xi(\pi n_1/l_2 - \xi'/\eta)} d\xi$$

As a result of these transformations, we obtain

$$\psi(\xi') = \frac{1}{2i\sqrt{2\pi}} \sqrt{\frac{2}{l_2}} \left(\int_0^{l_2} e^{i\xi(\pi n_1/l_2 + \xi'/\eta)} d\xi - \int_0^{l_2} e^{-i\xi(\pi n_1/l_2 - \xi'/\eta)} d\xi \right) \tag{A.2.4}$$

Let's calculate the first integral in (A.2.4)

$$\int_{-\infty}^{\infty} e^{i\xi(\pi n_1\xi/l_2+\xi'/\eta)}d\xi = \frac{\int_0^{l_2} e^{i\xi(\pi n_1/l_2+\xi'/\eta)}d[i\xi(\pi n_1/l_2+\xi'/\eta)]}{i(\pi n_1/l_2+\xi'/\eta)}$$

$$\frac{\int_0^{l_2} e^{i\xi(\pi n_1\xi/l_2+\xi'/\eta)}di\xi(\pi n_1/l_2+\xi'/\eta)}{i(\pi n_1/l_2+\xi'/\eta)} = \left.\frac{e^{i\xi(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)} - \frac{e^{i0(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)} - \frac{1}{i(\pi n_1/l_2+\xi'/\eta)}$$

As a result of these calculations, we obtain

$$\left.\frac{e^{i\xi(\pi n_1/l_2+\xi'/\eta)}}{i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2+\xi'/\eta)}-1}{i(\pi n_1/l_2+\xi'/\eta)} \qquad \text{(A.2.5)}$$

Let's calculate the second integral in (A.2.4)

$$\int_0^{l_2} e^{-i\xi(\pi n_1\xi/l_2-\xi'/\eta)}d\xi = \frac{\int_0^{l_2} e^{-i\xi(\pi n_1/l_2-\xi'/\eta)}d[-i\xi(\pi n_1/l_2-\xi'/\eta)]}{-i(\pi n_1/l_2-\xi'/\eta)}$$

$$\frac{\int_0^{l_2} e^{-i\xi(\pi n_1/l_2-\xi'/\eta)}d-i\xi(\pi n_1/l_2-\xi'/\eta)}{-i(\pi n_1/l_2-\xi'/\eta)} = \left.\frac{e^{-i\xi(\pi n_1\xi/l_2-\xi'/\eta)}}{-i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2}$$

$$\left.\frac{e^{-i\xi(\pi n_1/l_2-\xi'/\eta)}}{-i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{-il_2(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)} - \frac{1}{i(\pi n_1/l_2-\xi'/\eta)}$$

As a result of these calculations, we obtain

$$\left.\frac{e^{-i\xi(\pi n_1/l_2-\xi'/\eta)}}{-i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{-il_2(\pi n_1/l_2-\xi'/\eta)}-1}{-i(\pi n_1/l_2-\xi'/\eta)} \qquad \text{(A.2.6)}$$

Substituting (A.2.5) and (A.2.6) and (A.2.4), we have

$$\psi(\xi') = \frac{1}{2i\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{il_2(\pi n_1/l_2+\xi'/\eta)}-1}{i(\pi n_1/l_2+\xi'/\eta)} - \frac{e^{-il_2(\pi n_1/l_2-\xi'/\eta)}-1}{-i(\pi n_1/l_2-\xi'/\eta)}\right)$$

Let's do the transformations

$$\psi(\xi') = \frac{1}{2ii\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right)$$

$$\psi(\xi') = -\frac{1}{2\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right)$$

Finally we get the result of integration (A.2.1)

$$\psi(\xi') = -\sqrt{\frac{1}{4\pi\, l_2}}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right) \qquad \text{(A.2.7)}$$

Similarly, we take the integral (A.2.2)

$$\psi*(\xi') = \frac{1}{\sqrt{2\pi}}\int_0^{l_2}\sqrt{\frac{2}{l_2}}\sin(\pi n_1\xi/l_2)\exp\{-i\xi'\xi/\eta\}d\xi$$

Let's represent (A.2.2) in the form

$$\psi*(\xi') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_0^{l_2}\frac{e^{i\pi n_1\xi/l_2}-e^{-i\pi n_1\xi/l_2}}{2i}e^{-i\xi'\xi/\eta}d\xi$$

$$\psi*(\xi') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_0^{l_2}\frac{e^{i\pi n_1\xi/l_2}e^{-i\xi'\xi/\eta}-e^{-i\pi n_1\xi/l_2}e^{-i\xi'\xi/\eta}}{2i}d\xi$$

$$\psi*(\xi') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_{-\infty}^{\infty}\frac{e^{i\pi n_1\xi/l_2-i\xi'\xi/\eta}-e^{-i\pi n_1\xi/l_2-i\xi'\xi/\eta}}{2i}d\xi$$

$$\psi*(\xi') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_0^{l_2}\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}-e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}}{2i}d\xi$$

$$\psi*(\xi') = \frac{1}{2i\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\int_0^{l_2}e^{i\xi(\pi n_1/l_2-\xi'/\eta)}-e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}d\xi$$

As a result of these transformations, we obtain

$$\psi*(\xi') = \frac{1}{2i\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\int_0^{l_2}e^{i\xi(\pi n_1/l_2-\xi'/\eta)}d\xi-\int_0^{l_2}e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}d\xi\right) \qquad \text{(A.2.8)}$$

We calculate the first integral in (A.2.8)

$$\int_{-\infty}^{\infty} e^{i\xi(\pi n_1 \xi/l_2+\xi'/\eta)}d\xi = \frac{\int_0^{l_2} e^{i\xi(\pi n_1/l_2-\xi'/\eta)}d[i\xi(\pi n_1/l_2-\xi'/\eta)]}{i(\pi n_1\xi/l_2-\xi'/\eta)}$$

$$\frac{\int_0^{l_2} e^{i\xi(\pi n_1\xi/l_2-\xi'/\eta)}di\xi(\pi n_1/l_2-\xi'/\eta)}{i(\pi n_1/l_2-\xi'/\eta)} = \left.\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)} - \frac{e^{i0(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)} - \frac{1}{i(\pi n_1/l_2-\xi'/\eta)}$$

$$\left.\frac{e^{i\xi(\pi n_1/l_2-\xi'/\eta)}}{i(\pi n_1/l_2-\xi'/\eta)}\right|_0^{l_2} = \frac{e^{il_2(\pi n_1/l_2-\xi'/\eta)}-1}{i(\pi n_1/l_2-\xi'/\eta)} \qquad \text{(A.2.9)}$$

We calculate the second integral in (A.2.8)

$$\int_0^{l_2} e^{-i\xi(\pi n_1 \xi/l_2+\xi'/\eta)}d\xi = \frac{\int_0^{l_2} e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}d[-i\xi(\pi n_1/l_2+\xi'/\eta)]}{-i(\pi n_1/l_2+\xi'/\eta)}$$

$$\frac{\int_0^{l_2} e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}d-i\xi(\pi n_1/l_2+\xi'/\eta)}{-i(\pi n_1/l_2+\xi'/\eta)} = \left.\frac{e^{-i\xi(\pi n_1\xi/l_2+\xi'/\eta)}}{-i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2}$$

$$\left.\frac{e^{-i\xi(\pi n_1/l_2+\xi'/\eta)}}{-i(\pi n_1/l_2+\xi'/\eta)}\right|_0^{l_2} = \frac{e^{-il_2(\pi n_1/l_2+\xi'/\eta)}-1}{-i(\pi n_1/l_2+\xi'/\eta)} \qquad \text{(A.2.10)}$$

Substituting (A.2.5) and (A.2.6) and (A.2.4), we have

$$\psi^*(\xi') = \frac{1}{2i\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{il_2(\pi n_1/l_2-\xi'/\eta)}-1}{i(\pi n_1/l_2-\xi'/\eta)} - \frac{e^{-il_2(\pi n_1/l_2+\xi'/\eta)}-1}{-i(\pi n_1/l_2+\xi'/\eta)}\right)$$

Let's do the transformations

$$\psi^*(\xi') = \frac{1}{2ii\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)} + \frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\right)$$

$$\psi^*(\xi') = -\frac{1}{2\sqrt{2\pi}}\sqrt{\frac{2}{l_2}}\left(\frac{e^{i(\pi n_1 - \xi' l_2/\eta)} - 1}{(\pi n_1 / l_2 - \xi'/\eta)} + \frac{e^{-i(\pi n_1 + \xi' l_2/\eta)} - 1}{(\pi n_1 / l_2 + \xi'/\eta)}\right)$$

Finally, we obtain the result of integration (A.2.2)

$$\psi^*(\xi') = -\sqrt{\frac{1}{4\pi l_2}}\left(\frac{e^{i(\pi n_1 - \xi' l_2/\eta)} - 1}{(\pi n_1 / l_2 - \xi'/\eta)} + \frac{e^{-i(\pi n_1 + \xi' l_2/\eta)} - 1}{(\pi n_1 / l_2 + \xi'/\eta)}\right) \tag{A.2.11}$$

So, the results of taking the integrals (A.2.1) and (A.2.2) are the expressions (A.2.7) and (A.2.11):

$$\psi(\xi') = -\sqrt{\frac{1}{4\pi l_2}}\left(\frac{e^{i(\pi n_1 + \xi' l_2/\eta)} - 1}{(\pi n_1 / l_2 + \xi'/\eta)} + \frac{e^{-i(\pi n_1 - \xi' l_2/\eta)} - 1}{(\pi n_1 / l_2 - \xi'/\eta)}\right) \tag{A.2.12}$$

$$\psi^*(\xi') = -\sqrt{\frac{1}{4\pi l_2}}\left(\frac{e^{i(\pi n_1 - \xi' l_2/\eta)} - 1}{(\pi n_1 / l_2 - \xi'/\eta)} + \frac{e^{-i(\pi n_1 + \xi' l_2/\eta)} - 1}{(\pi n_1 / l_2 + \xi'/\eta)}\right) \tag{A.2.13}$$