Appendix 3

Product of expressions (2.74) and (2.75) {or (A.2.12) and (A.2.13)}

$$\psi(\xi')=-\sqrt{\frac{1}{4\pi l_2}}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right) \tag{A.3.1}$$

$$\psi^*(\xi')=-\sqrt{\frac{1}{4\pi l_2}}\left(\frac{e^{i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}+\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\right) \tag{A.3.2}$$

equally

$$p(\xi')=\psi(\xi')\psi^*(\xi')=\frac{1}{4\pi l_2}\left(\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}+\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\right)\left(\frac{e^{i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}+\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\right) \tag{A.3.3}$$

Opening large brackets, we multiply the terms in pairs

$$\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\frac{e^{i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}=\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{i(\pi n_1-\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}$$

$$\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}=\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2+\xi'/\eta\right)^2}$$

$$\frac{e^{i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}=\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2+\xi'/\eta\right)^2}$$

$$\frac{e^{-i(\pi n_1-\xi' l_2/\eta)}-1}{(\pi n_1/l_2-\xi'/\eta)}\frac{e^{-i(\pi n_1+\xi' l_2/\eta)}-1}{(\pi n_1/l_2+\xi'/\eta)}=\frac{\left(e^{-i(\pi n_1-\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}$$

Add the resulting expressions

$$\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{i(\pi n_1-\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}+\frac{2\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2+\xi'/\eta\right)^2}+\frac{\left(e^{-i(\pi n_1-\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}$$

Rearranging terms and summing up them, we get

$$\frac{\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{i(\pi n_1-\xi' l_2/\eta)}-1\right)+\left(e^{-i(\pi n_1-\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}+\frac{2\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)}{\left(\pi n_1/l_2+\xi'/\eta\right)^2}$$

(A.3.4)

Performing calculations

1. $\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{i(\pi n_1-\xi' l_2/\eta)}-1\right)=e^{i2\pi n_1}-e^{i(\pi n_1+\xi' l_2/\eta)}-e^{i(\pi n_1-\xi' l_2/\eta)}+1$

2. $\left(e^{-i(\pi n_1-\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)=e^{-i2\pi n_1}-e^{-i(\pi n_1-\xi' l_2/\eta)}-e^{-i(\pi n_1+\xi' l_2/\eta)}+1$

3. $\left(e^{i(\pi n_1+\xi' l_2/\eta)}-1\right)\left(e^{-i(\pi n_1+\xi' l_2/\eta)}-1\right)=e^{0}-e^{i(\pi n_1+\xi' l_2/\eta)}-e^{-i(\pi n_1+\xi' l_2/\eta)}+1=$

$$=1-(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})+1=-(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})+2=$$

$$-2[(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})/2-1]=\ -2[\cos(\pi n_1+\xi' l_2/\eta)-1] \qquad \text{(A.3.5)}$$

where the expression $\cos x=\dfrac{e^{ix}+e^{-ix}}{2}$ is taken into account.

Add 1 and 2

$$e^{i2\pi n_1}-e^{i(\pi n_1+\xi' l_2/\eta)}-e^{i(\pi n_1-\xi' l_2/\eta)}+e^{-i2\pi n_1}-e^{-i(\pi n_1-\xi' l_2/\eta)}-e^{-i(\pi n_1+\xi' l_2/\eta)}+2$$

Let’s regroup the terms

$$(e^{i2\pi n_1}+e^{-i2\pi n_1})-(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})-(e^{i(\pi n_1-\xi' l_2/\eta)}+e^{-i(\pi n_1-\xi' l_2/\eta)})+2$$

or

$$2[(e^{i2\pi n_1}+e^{-i2\pi n_1})/2-(e^{i(\pi n_1+\xi' l_2/\eta)}+e^{-i(\pi n_1+\xi' l_2/\eta)})/2-(e^{i(\pi n_1-\xi' l_2/\eta)}+e^{-i(\pi n_1-\xi' l_2/\eta)})/2+1]=$$

$$=\ 2[\cos 2\pi n_1-\cos(\pi n_1+\xi' l_2/\eta)-\cos(\pi n_1-\xi' l_2/\eta)+1] \qquad \text{(A.3.6)}$$

Let’s substitute the terms (A.3.5) and (A.3.6) into (A.3.4), we obtain

$$\frac{2[(\cos 2\pi n_1-\cos(\pi n_1+\xi' l_2/\eta)-\cos(\pi n_1-\xi' l_2/\eta)+1]}{\left(\pi n_1/l_2\right)^2-\left(\xi'/\eta\right)^2}-\frac{4[\cos(\pi n_1+\xi' l_2/\eta)-1]}{\left(\pi n_1/l_2+\xi'/\eta\right)^2} \qquad \text{(A.3.7)}$$

Now insert (A.3.7) into (A.3.3)

$$p(\xi') = \frac{1}{4\pi l_2}\left(\frac{2[(\cos 2\pi n_1 - \cos(\pi n_1 + \xi' l_2/\eta) - \cos(\pi n_1 - \xi' l_2/\eta) + 1]}{\left(\pi n_1 / l_2\right)^2 - \left(\xi' / \eta\right)^2} - \frac{4[\cos(\pi n_1 + \xi' l_2/\eta) - 1]}{\left(\pi n_1 / l_2 + \xi'/\eta\right)^2} \right) \tag{A.3.8}$$

We use two trigonometric formulas

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad \text{и} \quad \cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)] \tag{A.3.9}$$

Where should

$$\cos 2\pi n_1 + 1 = 2\cos^2 \pi n_1 \tag{A.3.10}$$

$$\cos(\pi n_1 - \xi' l_2/\eta) + \cos(\pi n_1 + \xi' l_2/\eta) = 2\cos(\pi n_1)\cos(\xi' l_2/\eta) \tag{A.3.11}$$

In view of (A.3.10) and (A.3.11), the expression (A.3.8) takes the form

$$p(\xi') = \frac{1}{4\pi l_2}\left(\frac{2[2\cos^2 \pi n_1 - 2\cos(\pi n_1)\cos(\xi' l_2/\eta)]}{\left(\pi n_1 / l_2\right)^2 - \left(\xi' / \eta\right)^2} - \frac{4[\cos(\pi n_1 + \xi' l_2/\eta) - 1]}{\left(\pi n_1 / l_2 + \xi'/\eta\right)^2} \right)$$

Performing simplifications

$$p(\xi') = \frac{1}{4\pi l_2}\left(\frac{4[\cos^2 \pi n_1 - \cos(\pi n_1)\cos(\xi' l_2/\eta)]}{\left(\pi n_1 / l_2\right)^2 - \left(\xi' / \eta\right)^2} - \frac{4[\cos(\pi n_1 + \xi' l_2/\eta) - 1]}{\left(\pi n_1 / l_2 + \xi'/\eta\right)^2} \right)$$

finally get

$$p(\xi') = \frac{1}{\pi l_2}\left(\frac{[\cos^2 \pi n_1 - \cos(\pi n_1)\cos(\xi' l_2/\eta)]}{\left(\pi n_1 / l_2\right)^2 - \left(\xi' / \eta\right)^2} - \frac{[\cos(\pi n_1 + \xi' l_2/\eta) - 1]}{\left(\pi n_1 / l_2 + \xi'/\eta\right)^2} \right) \tag{A.3.12}$$