3 Derivation of Schrödinger's equation

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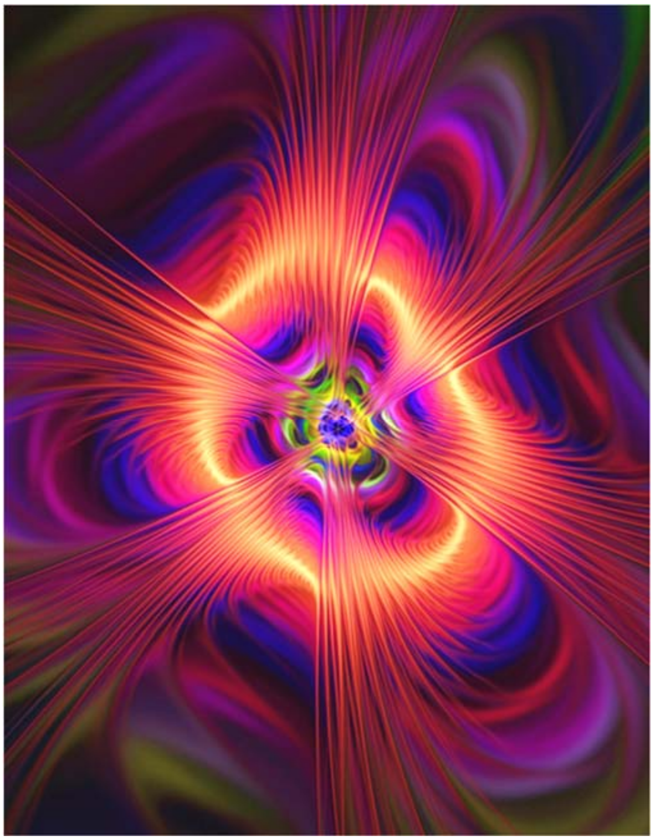

The base concepts of quantum mechanics (such as the matter waves of de Broglie, the Heisenberg Uncertainty Principle, the lack of definite size and trajectory of elementary particles, the universality of Planck's constant and Schrödinger's equation) still lack sufficient logical justification. The interest in the origins of quantum mechanics is caused, among others, by the fact that the cutting edge of science in the study of structural organization of matter, string theory, is based on quantum mechanics. Yet there are, in my opinion, virtually insurmountable difficulties in this theory. This makes it necessary to rethink the foundations of quantum physics.

In this Chapter, a model of a material particle in chaotic motion (while maintaining its size and trajectory) is presented. On the basis of this model, the following is achieved:

- ❖ to express Planck's constant h through the main features of a stationary random process;
- ❖ to justify the transition from the coordinate representation of the state of the particle to its momentum representation without invoking the idea of the existence of de Broglie waves or the Heisenberg uncertainty principle.
- ❖ to derive Schrödinger equation on the basis of the principle of extremum of the mean of the action of a particle in chaotic motion.

At the same time, the conditions and limits of application of the generalized Schrödinger equation to describe phenomena on microscopic and macroscopic scales are highlighted.

An intermediate result, the determination of the density of the probability distribution of the n-th derivative of an n^{th}-order differentiable, random, stationary process, can be applied in many areas of probability theory and statistical physics.

Methods: In deriving the generalized Schrödinger equation were applied: the methods of probability theory, the theory of stochastic processes, the theory of generalized functions, and calculus of variations. The formalism of quantum mechanics was also taken into account.

3.1 A brief history of the emergence of Schrödinger equation

The idea conceived by Louie de Broglie that material particles could possess wave properties was of particular importance in the 1920's. In his doctoral thesis, *Recherches sur la théorie des quanta* (Research on the Theory of the Quanta, 1924), Louis de Broglie compared the rectilinear trajectory of the free motion of a particle with a direct ray of light, and came to the conclusion that they are described by the same Jacobi equation, arising from the fundamental principle of "extremum of action". It turned out that the trajectory of the free motion of the particle and the beam of light are extrema for virtually the same functional of the action. This circumstance prompted Louis de Broglie to suggest that if the wave described by the equation

Louie de Broglie

$$w = exp\{i(\omega t - \mathbf{kr})\}, \tag{3.1}$$

where ω is the angular frequency; $\mathbf{k}$ is the propagation vector; t is time; $\mathbf{r}$ is the dimensional vector, displays some properties of a particle. The opposite assertion is quite possible that is a moving material particle can correspond to a plane wave described by

$$\psi = exp\{i(Et - \mathbf{pr})/\hbar\}, \tag{3.2}$$

where E is the kinetic energy of a moving particle, $\mathbf{p} = m\mathbf{v}$ is its momentum, $\hbar$ is the Dirac constant (or reduced Planck constant) associated with the Planck constant by the relation $\hbar = h/2\pi$.

In addition, Louis de Broglie was acquainted with experiments, carried out by his elder brother Maurice de Broglie, which were associated with the physics of X-ray radiation, as well as with the pioneering work of Max Planck and Albert Einstein on the quantum nature of radiation and absorption of light. This allowed him in 1923 to 1924 to propose that a moving particle can be associated with an oscillatory perturbation ψ having frequency

$$\omega = E/\hbar \tag{3.3}$$

and with the wavelength

$$\lambda = 2\pi\hbar/|\mathbf{p}|. \tag{3.4}$$

This idea was supported by P. Langevin and A. Einstein, but most of the physics community reacted to it with skepticism. However, in the period from 1927 to 1930, several groups of experimenters (C. Davisson & L. Germer, and O. Stern, I. Estermann et al.) showed that the idea of the existence of matter waves, proposed by de Broglie, could be used to describe the phenomenon of the diffraction of electrons and atoms in crystals.

Erwin Schrödinger

In one of his early works of 1925 to 1926, Erwin Schrödinger, critical of the Bose-Einstein statistics formulation, wondered: "Why not start with the wave representation of the gas particles, and then impose on such ‘waves’ the quantization conditions ‘à la the Debye condition’?" After that followed his central idea: "This implies none other than the need to take seriously into consideration the proposal of L. de Broglie and A. Einstein concerning the wave theory of moving particles."

This idea served as one of the reasons that Schrödinger found the equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U \psi , \qquad (3.5)$$

where $\psi = \psi(x,y,z,t)$ is the wave function describing the state of an elementary particle, U is the potential energy of the particle, and m is the mass of the particle.

Equation (5) laid the foundation for the intensive development of quantum mechanics, together with the great works of Max Planck, Albert Einstein, Niels Bohr, and Werner Heisenberg. However, the arguments presented by Schrödinger in deriving equation (3.5) were subsequently recognized by experts as incorrect, but the equation itself turned out to be correct.

This was not the only such case in science. For example, the fundamental equations of electrodynamics were derived by James Clerk Maxwell from incorrect assumptions about the mechanical properties of the ether.

Over the past ninety-five years since 1926, many researchers have proposed different ways to derive the Schrödinger equation (5) based on the axioms of many different interpretations of quantum mechanics (see for example [56 to 66]). But dissatisfaction in understanding the logical foundations of quantum physics remains to this day. The situation is so complicated that David Mermin suggested leaving "unnecessary disputes" and simply "Shut up and calculate!"

Nevertheless, in this Chapter it is proposed to make one more attempt to think first and then to calculate, that is, "Think and calculate".

The probabilistic model of a randomly wandering particle (which has a volume and a continuous trajectory of motion) considered in this article clearly contradicts almost all modern interpretations of quantum mechanics, but this probabilistic model also leads to a derivation of the Schrödinger equation, as indicated below.

3.2 Probabilistic model of a particle moving along a chaotic trajectory

Consider a particle occupying a small volume compared to that of its surrounding space (see Figure 3.1). Conventionally, call this particle a "point".

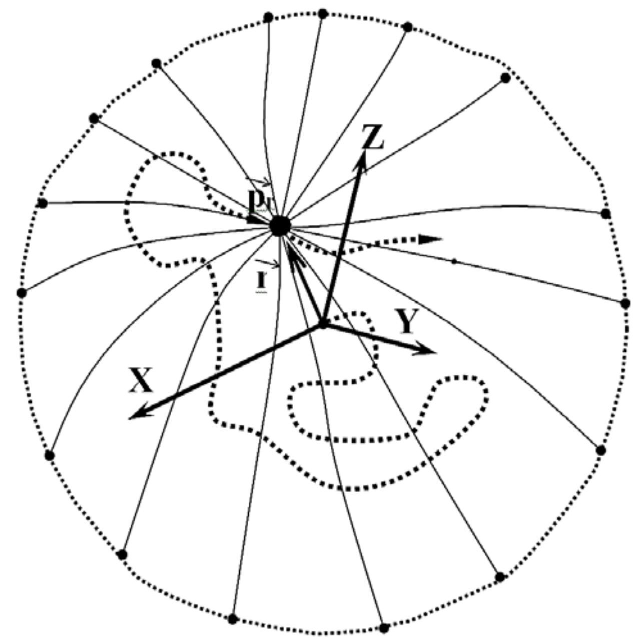

Fig. 3.1. The particle ("point") in chaotic motion in the neighborhood of the "center" of the coordinate system X Y Z, so that its total mechanical energy *E* always remains constant.

Suppose that this "point" constantly chaotic motion around the conditional "center" (combined with the origin of the coordinate system XYZ) under the influence of various mutually independent force factors. Examples of such a "point" in continuous chaotic motion may be: an atom vibrating in a crystal lattice; a fly flying in a jar; a nuclei vibrating inside a biological cell, a human embryo moving in the maternal womb; a tip of branch fluttering in the wind, and so forth.

We suppose that such chaotic motion of the "point" continues "forever" due to the fact that its total mechanical energy E always remains constant:

$$E = T\,(x,y,z,t) \;+\; U\,(x,y,z,t) = const, \qquad (3.6)$$

where $T(x,y,z,t)$ is the kinetic energy of the "point" due to its velocity, and $U(x,y,z,t)$ is the potential energy of the "point" associated with the force tending to return it to the "center" of the coordinate system XYZ (Figure 3.1).

Thus, in this model, each of the energies $T(x,y,z,t)$ and $U(x,y,z,t)$ of the "point" is a random function of time and its position relative to the "center". But these energies flow smoothly into each other so that their sum (i.e., the total mechanical energy E) always remains constant.

If the speed of the "point" in chaotic motion in the vicinity of the “center” (Figure 3.1) is low, then according to non-relativistic mechanics, it has kinetic energy

$$T(x,y,z,t) = \frac{p_x^2(x,y,z,t) + p_y^2(x,y,z,t) + p_z^2(x,y,z,t)}{2m}. \quad (3.7)$$

For brevity, instead of (3.7) we write

$$T(t) = \frac{p_x^2(t) + p_y^2(t) + p_z^2(t)}{2m}, \quad (3.8)$$

where $p_x(t)$, $p_y(t)$, $p_z(t)$ are the respective instantaneous values of the spatial components of the momentum of the "point" in chaotic motion; m is the mass of the "point".

Wherein

$$|\vec{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}, \quad (3.9)$$

where

$$p_i = mv_i = m\frac{dx_i}{dt} = m \cdot x_i'. \quad (3.10)$$

The type of potential energy $U(x,y,z,t)$ acting on the “point” is not specified.

The action of the "point” S under consideration is defined in non-relativistic mechanics as follows [67]

$$S(t) = \int_{t_1}^{t_2} [T(p_x,t) - U(x,t)]dt + Et. \quad (3.11)$$

To simplify the calculations, let's consider the one-dimensional case, without loss of generality. The three-dimensional case merely requires more integrations.

Due to the complexity of the path of the “point" in motion, we are interested not in the action itself (3.11), but rather its average over time (resp., over its realizations).

Due to the complexity of the movement of the wandering “point”, we are interested not in the action itself (3.11), but in its average over realizations or over time

$$\overline{S} = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} S_i(t) = \int_{t_1}^{t_2}[\overline{T(p_x,t)} - \overline{U(x,t)}dt + \overline{E}t. \quad (3.12)$$

Recall that for an ergodic stochastic process, an average over time is equivalent to the average over its realizations.

Finding the mean of the action (3.12) is carried out over the realizations, taken for the same time interval

$$\Delta t = t_2 - t_1.$$

The mean kinetic energy of a "point" in chaotic motion may be represented as

$$\overline{T(p_x,t)} = \frac{1}{2m}\int_{-\infty}^{\infty}\rho(p_x)p_x^2 dp_x\ , \qquad (3.13)$$

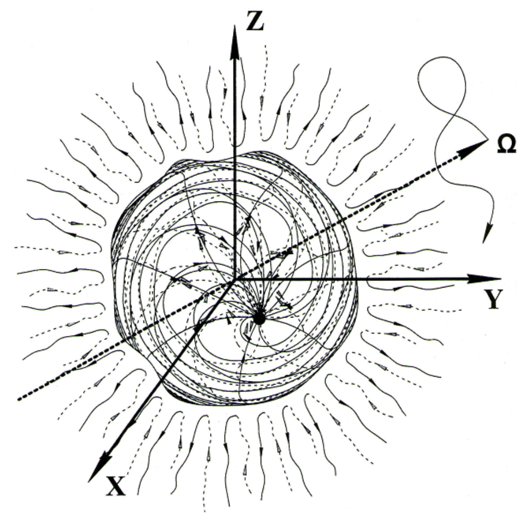

Fig. 2. In the medium, a spherically symmetric formation, inside of which a particle (“point”) constantly wanders randomly

where $\rho(p_x)$ is the probability density function of the momentum component p_x of the "points".

The averaged potential energy of a "point" may be represented as

$$\overline{U(x,t)} = \int_{-\infty}^{\infty}\rho(x)U(x)dx, \qquad (3.14)$$

where $\rho(x)$ is the probability density function of the projection onto the x-axis of the “point” wandering in the vicinity of the conditional “center” (see Figures 3.1 and 3.2).

Substituting (3.13) and (3.14) into the equation (3.12) for the mean of the action, we obtain

$$\overline{S} = \int_{t_1}^{t_2} \left\{ \frac{1}{2m} \int_{-\infty}^{\infty} \rho(p_x) p_x^2 dp_x - \int_{-\infty}^{\infty} \rho(x) U(x) dx \right\} dt + \overline{E} t. \qquad (3.15)$$

For the further derivation of the Schrödinger equation, two auxiliary items are given below. The first item, developed by the author of this article, is dedicated to the definition of the probability density function of the derivative of the *n*-th order of the *n* times differentiable, stochastic stationary process. The second item, the "coordinate representation of the average momentum of a particle (i.e. “point”) is borrowed from the book of Blokhintzev D.I. [13], since this paragraph is of great importance for the aim set in this work.

3.3 Determination of the probability density function of the *n*-th derivative of an *n*-times differentiable stationary stochastic process

The key to the understanding of quantum mechanics and the limits of its application lies in the determination of the probability density function of the derivative of a stationary stochastic process, given that the probability density function of the stochastic process itself is already known.

The solution to this problem to justify the quantum-mechanical procedure of the transition from the coordinate representation to the momentum representation, and vice versa, without using the hypothesis of the existence of de Broglie waves.

This is made possible due to the fact that the momentum of a particle (material “point”) is linearly related to the derivative of its coordinate: $p_x = m\, \partial x/\partial t = mx'$.

In addition, the problem of determining the one-dimensional probability density function $\rho[\xi^n(t)]$, the derivative of the *n*-th order *n*-times differentiable stationary stochastic process $\xi(t)$, when only its one-dimensional probability density function $\rho[\xi(t)]$ is known, arises in a series of problems in the fields of statistical mechanics and radio physics.

First, consider the general properties of the first derivative of the stationary stochastic process $\xi(t)$. To do this, let's explore its realizations (see Figure 3.3).

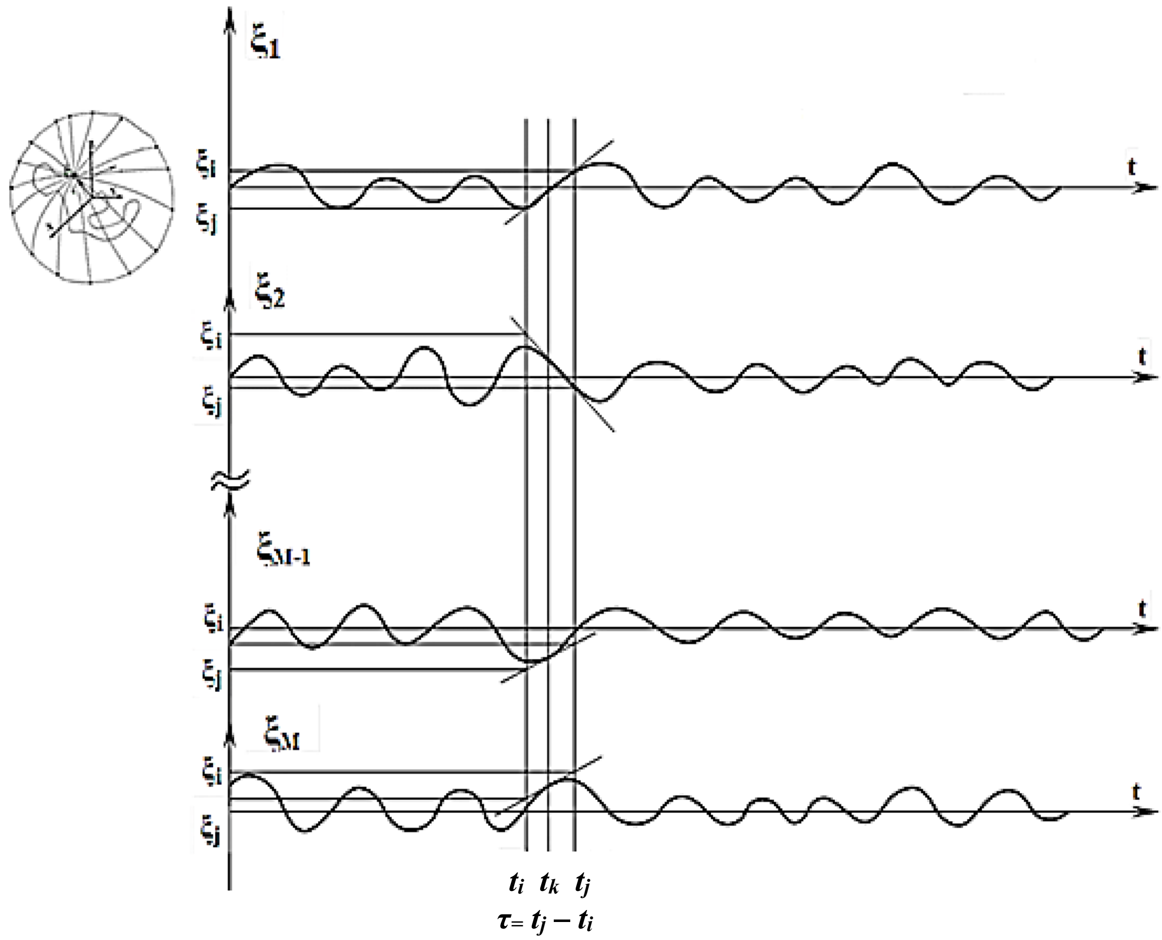

Fig. 3.3. Realizations of the differentiable stationary stochastic process $\xi(t)$. It is shown that the variables $\xi(t_k) = \xi_k$ in the cross section t_k and the derivative of the stationary random process in the same cross section $\xi'(t_k) = \xi_k'$ are independent random variables

Figure 3.3 shows that the value of the random variable $\xi(t_k)$ in the cross section t_k is independent from the derivative $\xi'(t_k) = \frac{\partial \xi(t_k)}{\partial t}$ of this process taken in the same cross section t_k,. Therefore, the random values $\xi(t_k)$ and $\xi'(t_k)$ are uncorrelated. This may be expressed analytically [45]:

$$< \xi(t_k)\xi'(t_k) > \; = \; < \frac{d}{dt}\frac{1}{2}[\xi(t_k)]^2 > \; = \; \frac{1}{2}\frac{d}{dt} < [\xi(t_k)]^2 > \; = \; 0, \qquad (16)$$

where the brackets < > means averaging over the realizations. Here it is taken into account that the differentiation and averaging operations in this case are commutative, and that all the averaged characteristics of a stationary (in the narrow sense) stochastic process, including its dispersion, are constant over time: $<[\xi(t_k)]^2> = const$.

Realizations of a stationary stochastic process $\xi(t)$, such as that shown in Figure 3, can be interpreted as the change over time of the projection onto the *X*-axis of the position of the wandering "point" in motion (see Figures 3.2 and 3.3), i.e. $x(t) = \xi(t)$.

However, even in the case of the statistical independence of the random values $\xi(t_k) = \xi_k$ and $\xi'(t_k) = \xi_k'$, there exists a connection between the probability density functions $\rho(\xi_k)$ and $\rho(\xi_k')$. This follows from the procedure of obtaining the probability density function of a derivative $\rho(\xi_k')$ for a known two-dimensional probability density function of a stationary stochastic process (Figures 3.3) [45, 46]

$$\rho(\xi_i, \xi_j) = \rho(\xi_i, t_i; \xi_i, t_j). \qquad (3.17)$$

For this, in expression (3.17), it is necessary to make a change of variables

$$\xi_i = \xi_k - \frac{\tau}{2}\xi_k'; \quad \xi_j = \xi_k + \frac{\tau}{2}\xi_k'; \quad t_i = t_k - \frac{\tau}{2}; \quad t_j = t_k + \frac{\tau}{2}, \qquad (3.18)$$

$$\text{where} \quad \tau = t_j - t_i; \quad t_k = \frac{t_j - t_i}{2}$$

with the Jacobian of the transformation $[J] = \tau$. As a result, from the probability density function (3.17) we obtain

$$\rho(\xi_k, \xi_k') = \lim_{\tau \to 0} \tau \rho\left(\xi_k - \frac{\tau}{2}\xi_k', t_k - \frac{\tau}{2}; \xi_k + \frac{\tau}{2}\xi_k', \; t_k + \frac{\tau}{2}\right). \qquad (3.19)$$

Further, integrating the obtained expression over ξ_k, find the desired probability density function of the derivative of the original process in the cross section t_k [45, 46]:

$$\rho(\xi'_k) = \int_{-\infty}^{\infty} \rho(\xi_k, \xi'_k)\, d\xi_k \, . \tag{3.20}$$

The formal procedure given by (3.17) through (3.20) solves the problem of determining the probability density function $\rho(\xi')$ for a known two-dimensional probability density function (3.17). However, a two-dimensional probability density function is defined only for a very limited class of stochastic processes. It is therefore necessary to consider the possibility of obtaining a probability density function $\rho(\xi')$ for a known one-dimensional probability density function $\rho(\xi)$.

To solve this problem, use the following properties of stochastic processes:

1] A two-dimensional probability density function of any stochastic process can be represented as [45, 46]

$$\rho(\xi_i, t_i; \xi_j, t_j) = \rho(\xi_i, t_i)\rho(\xi_j, t_j / \xi_i, t_i), \tag{3.21}$$

where $\rho(\xi_j, t_j / \xi_i, t_i)$ is conditional probability density function .

2] For the strictly stationary stochastic process, the following identity holds [45, 46]

$$\rho(\xi_i, t_i) = \rho(\xi_j, t_j) \tag{3.22}$$

3] The conditional probability density function $\rho(\xi_j, t_j / \xi_i, t_i)$ of a stationary stochastic process as t_i tends to t_j degenerates into a delta function [46]

$$\lim_{\tau \to 0} \rho(\xi_i, t_i / \xi_j, t_j) = \delta(\xi_i - \xi_j). \tag{3.23}$$

Based on the above properties, consider a stochastic process over the interval $]t_i = t_k - \tau/2;\ t_j = t_k + \tau/2[$ (see Figures 3.3) as τ tends to zero, using the following formal procedure. The probability density functions $\rho(\xi_i) = \rho(\xi_i, t_i)$ and $\rho(\xi_j) = \rho(\xi_j, t_j)$ can always be represented as the product of two functions:

$$\rho(\xi_i) = \varphi(\xi_i)\varphi(\xi_i) = \varphi^2(\xi_i), \tag{3.24}$$

$$\rho(\xi_j) = \varphi(\xi_j)\varphi(\xi_j) = \varphi^2(\xi_j),$$

where $\varphi(\xi_i)$ represents the wave function of a random variable ξ_i in the cross section t_i. (Figures 3.3);

$\varphi(\xi_j)$ represents the wave function of a random variable ξ_j in the cross section t_j.

For a strictly stationary stochastic process, we have the identity

$$\varphi(\xi_i) = \varphi(\xi_j), \tag{3.25}$$

as is easily seen by taking the square root of both sides of the identity (3.22). Then, according to (3.24), we obtain (3.25). Note that identity (3.25) is approximately true for the majority of non-stationary stochastic processes as τ tends to zero, that is,

$$\varphi(\xi_i, t_i) = \lim_{\tau \to 0} \varphi(\xi_j,\, t_j = t_i - \tau). \tag{3.26}$$

When the condition (3.25) holds, equation (3.21) can be represented in the symmetric form

$$\rho(\xi_i, \xi_j) = \varphi(\xi_i)\rho(\xi_j / \xi_i)\varphi(\xi_j), \tag{3.27}$$

where $\rho(\xi_j / \xi_i)$ is the conditional probability density function.

In expanded form (3.27) becomes

$$\begin{aligned} &\rho\left[\xi_i, t_i = t_k - \frac{\tau}{2}; \xi_j, t_j = t_k + \frac{\tau}{2}\right] = \\ &= \varphi\left[\xi_i, t_j = t_k - \frac{\tau}{2}\right]\rho\left[\xi_j, t_j = t_k + \frac{\tau}{2} \;/\; \xi_i, t_i = t_k - \frac{\tau}{2}\right]\varphi\left[\xi_i, t_i = t_k + \frac{\tau}{2}\right]. \end{aligned} \tag{3.28}$$

In (3.28), let τ tend to zero, but in such a way that the interval τ is uniformly shrinks at the time $t_k = (t_j - t_i)/2$, then, taking into account (3.23), from (3.27) we obtain

$$\lim_{\tau \to 0} \rho(\xi_i, \xi_j) = \lim_{\tau \to 0} \{\varphi(\xi_i)\rho(\xi_j / \xi_i)\varphi(\xi_j)\} = \varphi(\xi_{ik})\delta(\xi_{ik} - \xi_{ik})\varphi(\xi_{jk}), \tag{3.29}$$

where ξ_{ik} is the result of the stochastic value $\xi(t_i)$ tending to the stochastic value $\xi(t_k)$ on the left, while ξ_{jk} is the result of the stochastic value $\xi(t_j)$ tending to the stochastic value $\xi(t_k)$ on the right.

Integrating both sides of the expression (3.29) over ξ_{ik} and ξ_{jk}, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\xi_{ik})\delta(\xi_{jk}-\xi_{ik})\varphi(\xi_{jk})d\xi_{ik}d\xi_{jk}=1. \tag{3.30}$$

Expression (3.30) is a formal mathematical identity out of the theory of generalized functions, taking into account the properties of the delta-function (or δ-function). In order to assign the expression (3.30) a physical meaning, it is necessary to specify the specific type of the δ-function.

Therefore now determine the form of a δ-function for a Markov stochastic process. Consider a continuous stochastic Markov process which satisfies the Einstein-Fokker equation [15, 46]

$$\frac{\partial\rho(\xi_j/\xi_i)}{\partial t}=B\frac{\partial^2\rho(\xi_j/\xi_i)}{\partial\xi^2}\ , \tag{3.31}$$

where B is the diffusion coefficient. This parabolic differential equation has three solutions, one of which can be represented as [15, 46]:

$$\rho\left(\xi_j,t_j/\xi_i,t_i\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{iq(\xi_j-\xi_i)-q^2B(t_j-t_i)\}dq, \tag{3.32}$$

where q is the generalized parameter. As $t_j - t_i = \tau$ tends to zero ($\tau \to 0$), then from (3.32) we obtain one of the definitions of a δ-function

$$\lim_{\tau\to\pm 0}\rho(\xi_i/\xi_j)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{iq(\xi_{jk}-\xi_{ik})\}dq=\delta(\xi_j-\xi_i), \tag{3.33}$$

Since this result was obtained for the limiting case as τ tends to zero ($\tau \to 0$), it is not excluded that the δ-function (3.33) can correspond not only to a Markov stochastic process, but also to many other stationary stochastic processes.

Substituting the δ-function (3.33) into (3.30) yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\xi_{ik})\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{iq(\xi_{jk}-\xi_{ik})\}dq\right]\varphi(\xi_{jk})d\xi_{ik}\,d\xi_{jk}=1. \tag{3.34}$$

Changing the order of integration in (3.34), we obtain

$$\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\xi_{ik})\exp\{-iq\xi_{ik}\}d\xi_{ik}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\xi_{jk})\exp\{iq\xi_{jk})d\xi_{jk}\right]dq=1. \qquad (3.35)$$

According to (3.25), for a stationary stochastic process the condition $\varphi(\xi_{ik}) = \varphi(\xi_{jk})$ is satisfied, and also from the properties of the δ-function at $\tau = 0$ follows that $\xi_{ik} = \xi_{jk} = \xi_{k}$. Therefore, expression (3.35) takes the form

$$\int_{-\infty}^{\infty}\varphi(q)\varphi^{*}(q)dq=1, \qquad (3.36)$$

where

$$\varphi(q)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\xi_k)\exp\{-iq\xi_k\}d\xi_k, \qquad (3.37)$$

$$\varphi^{*}(q)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\xi_k)\exp\{iq\xi_k\}d\xi_k. \qquad (3.38)$$

The integrand $\varphi(q)\varphi^{*}(q)$ in (3.36) satisfies all the requirements of the probability density function $\rho(q)$ of the random variable q:

$$\rho(q)=\varphi(q)\varphi^{*}(q)=|\varphi(q)|^{2}. \qquad (3.39)$$

Now investigate the random variable q. First let's reconsider the solution (3.32). The result of the integration on the right side of this expression does not depend on the variable q, therefore it may be considered as a generalized frequency. However, both the physical statement of the problem as well as the mathematical formalism in the expression (3.32) impose on q the following restrictions:

1) The variable q must be stochastic.

2) The random variable q must characterize a stochastic process in the interval under investigation $]t_i = t_k - \tau/2;\ t_j = t_k + \tau/2[$ (see Figure 3) as τ tends to zero;

3) The random variable q, according to the mathematical notation on the right side of (3.32), must belong to the set of real numbers ($q \in R$), having the cardinality of the continuum; that is, it must have the possibility to take any value in the range $]-\infty, \infty[$;

All these requirements are satisfied by any of the following random variables associated with a stochastic process in the studied time interval τ:

$$\xi_i' = \frac{\partial \xi_k}{\partial t}, \ \xi_i'' = \frac{\partial^2 \xi_k}{\partial t}, \ \dots , \ \xi^{(n)}{}_i = \frac{\partial^n \xi_i}{\partial t^n}. \tag{3.40}$$

But these random variables do not equally characterize the process. Consider one of the realizations of the test process. The function $\xi(t)$ (see Figure 3.3) in the interval $\tau = t_j - t_i$ (for $\tau < \tau_{cor}$, where τ_{cor} is the correlation time of a stochastic process) may be expanded as a Maclaurin series:

$$\xi(t_j) = \xi(t_i) + \xi'(t_i)\tau + \frac{\xi''(t_i)}{2}\tau^2 + \dots + \frac{\xi^{(n)}}{n!}\tau^n + \dots \tag{3.41}$$

Rewrite the expression (3.41) in the form

$$\frac{\xi_j - \xi_i}{\tau} = \xi_i' + \frac{\xi_i''}{2!}\tau + \dots + \frac{\xi^{(n)}\tau^{n-1}}{n!} + \dots \tag{3.42}$$

were $\xi(t_i) = \xi_i$, $\xi(t_j) = \xi_j$.

As in (3.33), let τ tend to zero, whereby (3.42) reduces to the identity

$$\lim_{\tau \to 0} \frac{\xi_j - \xi_i}{\tau} = \xi_k' \ \text{ were } \ \xi_k = \xi(t_k) \text{(see Figure 3.3).} \tag{3.43}$$

In this way, the only random variable satisfying all the above-mentioned requirements in the interval under investigation $]t_i = t_k - \tau/2;\ t_j = t_k + \tau/2[$ as τ tends to zero is the first derivative of the original stochastic process ξ'_k in the cross section t_k. Therefore we may assume that the random variable q in (3.32) through (3.39) is directly proportional to ξ'_k, that is

$$q = \frac{\xi'_k}{\eta}, \tag{3.44}$$

where $1/\eta$ is the proportionality coefficient.

Substituting (3.44) into (3.36) through (3.39), obtain the following procedure as required for determining the probability density function of the derivative $\rho(\xi'_k)$ of a stationary stochastic (not only Markov) process $\xi(t)$ in the cross section t_k, for a given the one-dimensional probability density function $\rho(\xi_k)$ in the same cross section t_k.

1] Express the given one-dimensional probability density function $\rho(\xi)$ as the product of two wave functions $\varphi(\xi)$:

$$\rho(\xi) = \varphi(\xi)\varphi(\xi). \tag{3.45}$$

2] Two Fourier transforms are then carried out:

$$\varphi(\xi') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi) \exp\{i\xi'\xi/\eta\} d\xi, \tag{3.46}$$

$$\varphi*(\xi') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi) \exp\{-i\xi'\xi/\eta\} d\xi. \tag{3.47}$$

3] Finally, for any given the cross section t of a stationary stochastic (not only Markov) process get the desired derivative of the probability density function:

$$\rho(\xi') = \varphi(\xi')\varphi^*(\xi') = |\varphi(\xi')|^2. \tag{3.48}$$

As we have already remarked, the procedure given by (3.45) through (3.48) can be applied not only to the stationary Markov processes, but to many other stationary stochastic processes for which the δ-function in (3.30) takes the form (3.33).

To clarify the physical meaning of the proportionality coefficient $1/\eta$, we use a comparison with known results. This method is not mathematically perfect, but allows us to quite efficiently obtain a reliable result of practical importance.

Consider a stationary Gaussian stochastic process $\xi(t)$. Wherein, in each cross section of this process, the random variable ξ is distributed according to the Gaussian distribution:

$$\rho(\xi) = \frac{1}{\sqrt{2\pi\sigma_\xi^2}} \exp\left\{-(\xi - a_\xi)^2 / 2\sigma_\xi^2\right\}, \tag{3.49}$$

where σ_ξ^2 and a_ξ are the variance and expected value of the given process $\xi(t)$.

Subjecting the probability density function (3.49) to the sequence of operations (3.45) through (3.48), we obtain the probability density function of the derivative of the stochastic process under consideration:

$$\rho(\xi') = \frac{1}{\sqrt{2\pi\left[\eta / 2\sigma_\xi\right]^2}} \exp\left\{-\frac{\xi'^2}{2\left[\eta / 2\sigma_\xi\right]^2}\right\}, \tag{3.50}$$

On the other hand, using the well-known procedure (3.17) through (3.20) for a similar case, we obtain [45, 46]

$$\rho(\xi') = \frac{1}{\sqrt{2\pi\sigma_{\xi'}^2}} \exp\left\{-\xi'^2 / 2\sigma_{\xi'}^2\right\}, \tag{3.51}$$

where $\sigma_{\xi'} = \sigma_\xi / \tau_{cor}$, and τ_{cor} is the correlation time of the initial stochastic process $\xi(t)$.

Comparing expressions (3.50) and (3.51), we find that for

$$\eta = \frac{2\sigma_\xi^2}{\tau_{cor}} \tag{3.52}$$

these probability density function’s are completely the same.

Expression (3.52) was obtained for a Gaussian stochastic process, but σ_ξ is the standard deviation and τ_{cor} is correlation time are the main characteristics of any stationary random process. All other initial and central moments in the case of a non-Gaussian distribution of the random variable $\xi(t)$ will give a small (insignificant) contribution to expression (3.52); therefore, it can be stated with a high degree of certainty that expression (3.52) is applicable to a wide class of stationary stochastic processes.

It should be noted that in statistical physics and quantum mechanics, the transition from the co-ordinate representation of a function of an elementary particle state to its momentum representation is effected by a formal process almost completely analogous to the procedure (3.45) through (3.48). The difference is only in determining the proportionality coefficient $1/\eta$.

In quantum mechanics it is well known that if the projection onto the x-axis of the position of a free elementary particle (for example, an electron) is described by a Gaussian distribution [35]

$$\rho(x) = |\psi(x)|^2 = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left\{-\frac{x^2}{2\sigma_x^2}\right\}, \tag{3.53}$$

where σ_x is the standard deviation of the projections of the positions of an elementary particle onto the x-axis in the neighborhood of the mean (that is, the “center” of the system). Then, as the result of operations analogous to that of (3.45) through (3.48), it turns out that the probability density function of the momentum component p_x of an elementary particle is also Gaussian [35]

$$\rho(p_x) = |\psi(p_x)|^2 = \frac{1}{\sqrt{2\pi\sigma_{p_x}^2}} \exp\left\{-\frac{p_x^2}{2\sigma_{p_x}^2}\right\} \tag{3.54}$$

with the standard deviation

$$\sigma_{p_x} = \frac{\hbar}{2\sigma_x}, \tag{3.55}$$

where $\hbar$ = $1.055 \cdot 10^{-34}$ J/Hz is the reduced Planck constant (or Dirac constant), which is related to the Planck constant $h = 6.626\ 070\ 15 \cdot 10^{-34}$ J/Hz by the ratio $\hbar = h/2\pi$.

If now take into account that the momentum component of an elementary particle (e.g. an electron) p_x is equal to

$$p_x = m_e \frac{dx}{dt} = m_e x' , \tag{3.56}$$

where m_e is the electron rest mass, then taking into account (3.55), the probability density function (3.54) becomes

$$\rho(x') = \frac{1}{\sqrt{2\pi\left[\hbar/(m_e 2\sigma_x)\right]^2}} \exp\left\{-\frac{x'^2}{2\left[\hbar/(m_e 2\sigma_x)\right]^2}\right\} . \tag{3.57}$$

$$\rho(\xi') = \frac{1}{\sqrt{2\pi\left[\eta/2\sigma_\xi\right]^2}} \exp\left\{-\frac{\xi'^2}{2\left[\eta/2\sigma_\xi\right]^2}\right\} , \tag{3.50}$$

Comparing (3.50) to (3.57), while taking into consideration (3.52) and that $\xi' = x'$ and $\sigma_\xi = \sigma_x$, find that for the given case

$$\eta = \frac{2\sigma_x^2}{\tau_{ex}} = \frac{\hbar}{m_e} , \tag{3.58}$$

where $$\tau_{ex} = \frac{2 m_e \sigma_x^2}{\hbar} = \frac{2 \cdot 0{,}91 \cdot 10^{-30}}{1{,}055 \cdot 10^{-34}} \cdot \sigma_x^2 = 1{,}73 \cdot 10^4 \sigma_x^2 \tag{3.59}$$

is the correlation time of a stationary stochastic process, which is the result of the projection of the stochastic motion of the "point" (e.g., an electron) onto the x-axis near the stationary "center" of the system (see Figures 3.1 and 3.2).

From expression (3.58) it follows that the reduced Planck constant is not a fundamental physical constant, but a quantity expressed through the main averaged parameters of a stationary stochastic process

$$\hbar = \frac{2\sigma_{par,x}^{2} m}{\tau_{par,x}}, \tag{3.60}$$

where for a general case:

$\sigma_{par,x}$ is the standard deviation of the projection of a randomly moving particle (“point”) on the x-axis in the vicinity of the average value (i.e., the “center” of the system);

$\tau_{par,x}$ is the correlation time of a given stationary stochastic process;

m is the mass of the particle ("points").

For a wide range of applications, the expression (3.60) is in itself very important, as is the related ratio (3.52), which in the general case may conveniently be represented as follows:

$$\eta_{par} = \frac{2\sigma_{par,x}^{2}}{\tau_{par,x}} = \frac{\hbar}{m}, \text{ with a dimension of } (m^2/s). \tag{3.61}$$

Note the following interim conclusions:

1] The quantum-mechanical transition from the coordinate representation to the momentum representation is not only applicable to the processes in the world of elementary particles, but also to any stationary Markov stochastic processes (and probably many other stochastic processes), both in the microcosm and in the macrocosm. For example, a tree branch, constantly moving chaotically around its middle position (the point of reference serving as the "center") by the rapidly changing direction of wind gusts, behaves similarly to elementary particles in the "potential well". The fluctuations of these movements of the branch would also have a discrete (quantum) average set of states, depending on the intensity of the wind gusts. With weak wind gusts, the branch generally fluctuates near the central reference point, in a way that the position of its tip can be described by a Gaussian distribution. With more intense gusts of wind, the tip of the branch rotates on average in a circle; with even greater gusts of wind, its tip basically describes the figure eight, etc. Depending on the strength of the wind, the tip of a branch can on average describe a discrete set of Lissajous figures. In other words, the quantum-mechanical formalism is not an exclusive feature of the microcosm; it is also applicable to the statistical description of many stochastic processes of the macrocosm.

2] The algorithm (3.45) through (3.48) of the transition from the coordinate representation (i.e., probability density function) $\rho(\xi_i)$ to the momentum representation (i.e., probability density function) $\rho(m\xi'_i)$ and vice versa was obtained with the concrete form of the δ-function (3.33). It would be interesting to analyze what would be the result in the case of other types of δ-function.

3] On the basis of the foregoing, we can obtain the probability density function $\rho(\xi_i'')$ of the second derivative of a stochastic process $\xi''(t)$. In this case, we should consider not the stochastic process $\xi(t)$ itself, but its first derivative $\xi'(t) = \partial\xi(t)/\partial t$. Then the distribution of the second derivative can be determined by the same procedure, only then instead of $\rho(\xi_i)$ in (3.45) through (3.48) it is necessary to substitute $\rho(\xi'_i)$.

Analogously, we also may obtain the probability density function $\rho(\xi_i^{(n)})$ of any derivative of n-times differentiable stationary stochastic process with the help of the following recursive procedure:

$$\rho(\xi^{(n-1)}) = \varphi\left(\xi^{(n-1)}\right)\varphi\left(\xi^{(n-1)}\right); \tag{62}$$

$$\varphi\left(\xi^{(n)}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi^{(n-1)}) \exp\left\{-\frac{i\xi^{(n)}\xi^{(n-1)}}{\eta_n}\right\} d\xi^{(n-1)}\ ; \tag{63}$$

$$\varphi^*\left(\xi^{(n)}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi^{(n-1)}) \exp\left\{\frac{i\xi^{(n)}\xi^{(n-1)}}{\eta_n}\right\} d\xi^{(n-1)}\ ; \tag{64}$$

$$\rho(\xi^{(n)}) = \varphi\left(\xi^{(n)}\right)\varphi^*\left(\xi^{(n)}\right),$$

where

$$\eta_n = \frac{2\sigma^2_{\xi^{(n-1)}}}{\tau_{cor\,\xi^{(n-1)}}}, \tag{65}$$

where $\sigma^2_{\xi^{(n-1)}}$, $\tau_{cor\,\xi^{(n-1)}}$ are the variance and the correlation time, respectively, of the given $n-1$ times differentiable stationary stochastic process.

4] The procedure (3.45) through (3.48), is completely analogous to the quantum-mechanical transition from the coordinate representation of a quantum system to its momentum representation, obtained here on the basis of a study of realizations of an ordinary stochastic stationary process, i.e. without involving the phenomenological principles of wave-particle duality.

There is also no need to use the de Broglie hypothesis about the existence of matter waves to describe the diffraction of atoms and electrons on the crystal lattice. We refer to Section 2.9.6 in [20], where one will find the derivation of the formula to create a three-dimensional diagram of the scattering of particles on the surface of a multilayer periodic crystal

$$\rho(\nu,\omega/\vartheta,\gamma)=4\pi n_1^2 k_\kappa \frac{\sin^2[\pi n_1/2-k_\kappa\sqrt{(a^2+b^2)/c^2}/2]}{[(\pi n_1)^2-k_\kappa^2(a^2+b^2)/c^2]^2}\cdot\left|\frac{c(a'_\nu b'_\omega-a'_\omega b'_\nu)+c'_\nu(ba'_\omega-ab'_\omega)}{c^2\sqrt{a^2+b^2}}\right|, \quad (3.66)$$

where

$a=\cos\nu\cos\omega+\cos\vartheta\cos\gamma$, $b=\cos\nu\sin\omega+\cos\vartheta\sin\gamma$, $c=\sin\nu+\sin\vartheta$, $a'_\nu=-\sin\nu\cos\omega$,
$b'_\nu=-\sin\nu\sin\omega$, $c'_\nu=\cos\nu$, $a'_\omega=-\cos\nu\sin\omega$, $b'_\omega=\cos\nu\cos\omega$,

angles ϑ, γ, ω and ν are shown in Figure 3.4;

$$k_\kappa = r_{cor}\, n_1^{1/2}/\,(0{,}066\, l_1),$$

where:

l_1 is the thickness of one layer, i.e. one sinusoidal equipotential surface;

n_1 is the number of layers effectively involved in the scattering of the particles;

r_{cor} is the average radius of curvature of a sinusoidal equipotential surface.

For a single crystal, all the sinusoidal equipotential surfaces have the same, that is, r_{cor}. Therefore in this case r_{cor} signifies the effective cross section of the scattering of electrons by the atoms of a crystal.

The results of a calculation using (3.66) at an angle of incidence of particles on the surface of the crystal using the values $\vartheta = 45^\circ$, azimuth angle $\gamma = 0^\circ$, $r_{cor} = 0.0000000001 = 10^{-10}$ cm, $l_1 = 0.000000001 = 10^{-9}$ cm, $n_1 = 1940$ (layers) are shown in Figure 3.4.

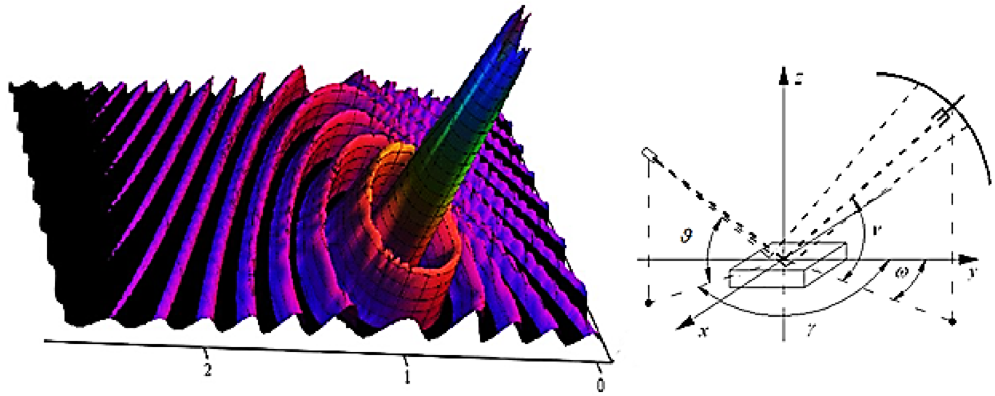

Fig. 3.4. Diagram of scattering of particles (electrons) on 1940 layers of sinusoidal equipotential crystal surfaces, calculated according to formula (3.66) using MathCad software [10, 20]

3.4 The coordinate representation of the average particle momentum

The contents of this paragraph are well known to specialists in the field of quantum mechanics. However, given the importance of subsequent conclusions and for ease of reference, the following calculations are almost completely rewritten from [13].

Let's first recall the properties of the Dirichlet integral appearing in the theory of Fourier integrals and the theory of generalized functions [13]

$$\lim_{k\to\infty}\frac{1}{\pi}\int_a^b \varphi(\tau)\frac{\sin k\tau}{\tau}d\tau = \begin{cases} 0, if \;\; a,b>0 \;\; or \;\; a,b<0, \\ \varphi(0), \; if \;\; a<0, \;\; b>0, \end{cases} \tag{3.67}$$

whereby

$$\lim_{k\to\infty}\frac{1}{\pi}\frac{\sin k\tau}{\tau} = \delta(\tau) \tag{3.68}$$

this is one of many forms of a δ-function.

Now consider the case of one dimension to reduce calculations and prove the equality [13]

$$\overline{p_x^n} = \int_{-\infty}^{+\infty}\rho(p_x)p_x^n dp_x = \int_{-\infty}^{+\infty}\psi(p_x)p_x^n\psi(p_x)dp_x = \int_{-\infty}^{+\infty}\psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)^n\psi(x)dx, \tag{3.69}$$

where n is a positive integer; $\overline{p_x^n}$ is averaging over time (or over implementations) of the momentum component raised to the power n

$$p_x^n = (m\cdot\partial x/\partial t)^n = (mx')^n, \tag{3.70}$$

where $\psi(x)$ *and* $\psi(p_x)$ are the wave functions (probability amplitude densities) which were introduced in (3.24) [$\psi(x)=\varphi(x)$] and (3.48) [$\psi(p_x)=\varphi(p_x)=\varphi(mx')$], and, according to (3.46) and (3.47), are related (provided the stationary stochastic process) by the Fourier transforms:

$$\psi(p_x = mx') = \int_{-\infty}^{+\infty}\psi(x)\frac{e^{i\frac{x'x}{\eta_{par}}}}{(2\pi)^{1/2}}dx = \int_{-\infty}^{+\infty}\psi(x)\frac{e^{i\frac{p_x x}{\hbar}}}{(2\pi\hbar)^{1/2}}dx\,; \tag{3.71}$$

$$\psi^*(p_x = mx') = \int_{-\infty}^{+\infty}\psi(x)\frac{e^{-i\frac{x'x}{\eta_{par}}}}{(2\pi)^{1/2}}dx = \int_{-\infty}^{+\infty}\psi(x)\frac{e^{-i\frac{p_x x}{\hbar}}}{(2\pi\hbar)^{1/2}}dx\,, \tag{3.72}$$

where the parameter η_{par} is defined by (3.61)

$$\eta_{par} = \frac{2\sigma^2_{par,x}}{\tau_{par,x}} = \frac{\hbar}{m}. \tag{3.73}$$

In order to prove statement (3.69), for $\psi(p_x)$ and $\psi^*(p_x)$ substitute their respective expressions in terms of integrals (3.71) and (3.72) [13]

$$\overline{p_x^n} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\psi(x_i)\frac{e^{-i\frac{p_x x_i}{\hbar}}}{(2\pi\hbar)^{1/2}}dx_i\, p_x^n\int_{-\infty}^{+\infty}\psi(x_j)\frac{e^{i\frac{p_x x_j}{\hbar}}}{(2\pi\hbar)^{1/2}}dx_j\right]dp_x. \tag{3.74}$$

A direct check makes it easy to verify that

$$p_x^n e^{i\frac{p_x x_j}{\hbar}} = \left(-i\hbar\frac{\partial}{\partial x_j}\right)^n e^{i\frac{p_x x_j}{\hbar}}. \tag{3.75}$$

Substituting (3.75) into (3.74), we obtain

$$\overline{p_x^n} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\psi*(x_k)e^{i\frac{p_x x_k}{\hbar}}dx_k \int_{-\infty}^{+\infty}\psi(x_l)\left(i\hbar\frac{\partial}{\partial x_l}\right)^n e^{-i\frac{p_x x_l}{\hbar}}dx_l\right]dp_x. \quad (3.76)$$

Let's perform an integration by parts n times, starting from the second integral in the integrand. In doing so, assume that $\psi(x)$ and its derivatives vanish at the integration boundaries, that is, at $x = \pm\infty$. Following these actions, we find [13]

$$\overline{p_x^n} = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\psi(x_i)e^{-i\frac{p_x x_i}{\hbar}}dx_i \int_{-\infty}^{+\infty}e^{i\frac{p_x x_j}{\hbar}}\left(-i\hbar\frac{\partial}{\partial x_j}\right)^n \psi(x_j)dx_j\right]dp_x. \quad (3.77)$$

Changing the order of integration, first integrate over p_x [13]

$$\overline{p_x^n} = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\psi(x_i)dx_i \int_{-\infty}^{+\infty}\left(-i\hbar\frac{\partial}{\partial x_j}\right)^n \psi(x_j)dx_j \int_{-\infty}^{+\infty}e^{i\frac{p_x(x_j-x_i)}{\hbar}}dp_x. \quad (3.78)$$

Introduce the variables $\xi = p_x/h$, $z = x_k - x_l$. In the last integral in (3.78), we perform the integration over ξ between the limits $-k$ to $+k$; then passing to the limit $k\to\infty$, this expression takes the form

$$\begin{aligned}\overline{p_x^n} &= \int_{-\infty}^{+\infty}\left[\left(-i\hbar\frac{\partial}{\partial x}\right)^n \psi(x)\right]dx\cdot\lim_{k\to\infty}\int_{-\infty}^{+\infty}\psi(x+\tau)\frac{\sin k\tau}{\pi\tau}d\tau = \\ &= \int_{-\infty}^{+\infty}\left[\left(-i\hbar\frac{\partial}{\partial x}\right)^n \psi(x)\right]dx\int_{-\infty}^{+\infty}\psi(x+\tau)\delta(\tau)d\tau.\end{aligned} \quad (3.79)$$

Based on the properties of the Dirichlet integral (3.67), when $a = -\infty$; $b = +\infty$ and $\psi(z) = \psi*(x+z)$, we have [13]

$$\overline{p_x^n} = \int_{-\infty}^{+\infty}\left[\left(-i\hbar\frac{\partial}{\partial x}\right)^n \psi(x)\right]\psi(x)dx = \int_{-\infty}^{+\infty}\psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)^n \psi(x)dx, \quad (3.80)$$

thus, expression (3.69) is proved [13].

It is easy to verify that expression (3.80) can be represented as

$$\overline{p_x^n} = \int_{-\infty}^{+\infty} \psi_t(x)\left(-i\hbar\frac{\partial}{\partial x}\right)^n \psi_t^*(x)dx, \tag{3.81}$$

where $$\psi(x,t) = \psi(x)\exp\{iut\}, \quad \psi*(x,t) = \psi(x)\exp\{-iut\}, \tag{3.82}$$

here u is an arbitrary real number.

Using (3.70) and (3.73) from the expression (3.81), we obtain

$$\overline{x'^n} = \overline{\left(\frac{dx}{dt}\right)^n} = \int_{-\infty}^{+\infty} \psi_t(x)\left(-i\eta_{par}\frac{\partial}{\partial x}\right)^n \psi_t^*(x)dx\ . \tag{3.83}$$

The generalization to three dimensions then increases the number of integrations.

3.5 Derivation of the Schrödinger equation

Let's return to the average action of the particle ("point") in chaotic motion (3.15)

$$\overline{S} = \int_{t_1}^{t_2}\left\{\frac{1}{2m}\int_{-\infty}^{\infty}\rho(p_x)p_x^2dp_x - \int_{-\infty}^{\infty}\rho(x)U(x)dx\right\}dt + \overline{E}t. \tag{3.84}$$

Imagine the action (3.84) in coordinate form. To do this, perform the following steps:

1] Writing the probability density function $\rho(x)$ as a product of two wave functions:

$$\rho(x) = \psi(x)\,\psi(x). \tag{3.85}$$

2] Let's use the coordinate representation of the averaged impulse raised to the n-th power (3.80). Wherein, in particular, for $n = 2$, we have

$$\overline{p_x^2} = \int_{-\infty}^{+\infty}\rho(p_x)p_z^2dp_x = \int_{-\infty}^{+\infty}\psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi(x)dx. \tag{3.86}$$

3] Using (3.86), we represent the averaged kinetic energy of the “point” (3.13) in the form

$$\overline{T} = \frac{1}{2m}\overline{p_x^2} = \frac{1}{2m}\int_{-\infty}^{\infty}\rho(p_x)p_x^2dp_x = \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\psi(x)\frac{\partial^2\psi(x)}{\partial x^2}dx. \tag{3.87}$$

4] By taking (3.85) into consideration, now represent the averaged potential energy of the "point" (3.14) in the form:

$$\overline{U} = \int_{-\infty}^{\infty} \psi(x) U(x) \psi(x) dx, \tag{3.88}$$

5] It is then easily seen that

$$E = \overline{E} = \int_{-\infty}^{\infty} i\hbar \psi(x) e^{iEt/\hbar} \frac{\partial \psi(x) e^{-iEt/\hbar}}{\partial t} dx = const. \tag{3.89}$$

6] Substituting expressions (3.87), (3.88) and (3.89) into (3.84), we obtain the record of the averaged action of a randomly wandering “point” ("point") in the coordinate form

$$\overline{S} = \int_{t_1}^{t_2} \left\{ \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \psi(x) \frac{\partial^2 \psi(x)}{\partial x^2} dx - \int_{-\infty}^{\infty} \psi(x) U(x) \psi(x) dx + \int_{-\infty}^{\infty} i\hbar \psi(x) e^{iEt/\hbar} \frac{\partial \psi(x) e^{-iEt/\hbar}}{\partial t} dx \right\} dt,$$

or $$\overline{S} = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \left(\frac{\hbar^2}{2m} \psi(x) \frac{\partial^2 \psi(x)}{\partial x^2} - \psi(x) U(x) \psi(x) + i\hbar \psi(x) e^{iEt/\hbar} \frac{\partial \psi(x) e^{-iEt/\hbar}}{\partial t} \right) dx dt. \tag{3.90}$$

Assuming that the function $\psi\,(x)$ varies with time [i.e. $\psi\,(x,t)$], but in such a way that at every instant it describes the stationary state of a random process, then expression (3.90) can be represented as:

$$\overline{S} = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \left(\frac{\hbar^2}{2m} \psi(x,t) \frac{\partial^2 \psi(x,t)}{\partial x^2} - \psi(x,t)^2 U(x) + i\hbar \psi(x,t) \frac{\partial \psi(x,t)}{\partial t} + E \psi(x,t)^2 \right) dx dt, \tag{3.91}$$

or

$$\overline{S} = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \left(\frac{\hbar^2}{2m} \psi(x,t) \frac{\partial^2 \psi(x,t)}{\partial x^2} + \psi(x,t)^2 [E - U(x)] + i\hbar \psi(x,t) \frac{\partial \psi(x,t)}{\partial t} \right) dx dt. \tag{3.92}$$

The extremality condition of the averaged action (3.92) requires the vanishing of its first variation [20]

$$\delta\overline{S} = \delta \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \left(\frac{\hbar^2}{2m} \psi(x,t) \frac{\partial^2 \psi(x,t)}{\partial x^2} + \psi(x,t)^2 [E - U(x)] + i\hbar \psi(x,t) \frac{\partial \psi(x,t)}{\partial t} \right) dx dt = 0. \tag{3.93}$$

Let’s find the extrema of functional (3.92), i.e., the function $\psi\,(x,t)$ for which the averaged action (3.92) takes an extreme value.

First, recall that an extrema of a functional of the form

$$S = \int L\left(x, t, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial t^2}, \frac{\partial^2 z}{\partial t \partial x} \right) dxdt \text{, где } z = \psi(x,t), \qquad (3.94)$$

defined by the Euler – Poisson equation [51]

$$L_z - \frac{\partial}{\partial x}\{L_p\} - \frac{\partial}{\partial t}\{L_g\} + \frac{\partial^2}{\partial x^2}\{L_r\} + \frac{\partial^2}{\partial t^2}\{L_t\} + \frac{\partial^2}{\partial x \partial t}\{L_s\} = 0 , \qquad (3.95)$$

where

L_z is the derivative of the Lagrangian L with respect to $z = \psi(x,t)$; (3.96)

L_p is the derivative of the L with respect to $p = \frac{\partial z}{\partial x} = \frac{\partial \psi(x,t)}{\partial x}$;

L_g is the derivative of the L with respect to $g = \frac{\partial z}{\partial t} = \frac{\partial \psi(x,t)}{\partial t}$;

L_r is the derivative of the L with respect to $r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 \psi(x,t)}{\partial x^2}$;

L_t is the derivative of the L with respect to $t = \frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 \psi(x,t)}{\partial t^2}$;

L_s is the derivative of the L with respect to $s = \frac{\partial^2 z}{\partial t \partial x} = \frac{\partial^2 \psi(x,t)}{\partial t \partial x}$,

wherein

$$\frac{\partial}{\partial x}\{L_p\} = L_{px} + L_{pz}\frac{\partial z}{\partial x} + L_{pp}\frac{\partial p}{\partial x} + L_{pg}\frac{\partial g}{\partial x} \qquad (3.97)$$

is the total partial derivative with respect to x;

$$\frac{\partial}{\partial t}\{L_g\} = L_{gt} + L_{gz}\frac{\partial z}{\partial t} + L_{gp}\frac{\partial p}{\partial t} + L_{gg}\frac{\partial g}{\partial t}$$

is the total partial derivative with respect to t;

$$\frac{\partial^2}{\partial x^2}\{L_r\} = \frac{\partial^2 L_r}{\partial x^2} + L_{rz}\frac{\partial z}{\partial x^2} + L_{rp}\frac{\partial p}{\partial x^2} + L_{rg}\frac{\partial g}{\partial x^2}$$

is the full second partial derivative with respect to x;

$$\frac{\partial^2}{\partial t^2}\{L_t\} = \frac{\partial^2 L_t}{\partial t^2} + L_{tz}\frac{\partial z}{\partial t^2} + L_{tp}\frac{\partial p}{\partial t^2} + L_{tg}\frac{\partial g}{\partial t^2}$$

is the full second partial derivative with respect to t;

$$\frac{\partial^2}{\partial t \partial x}\{L_s\} = \frac{\partial^2 L_s}{\partial t \partial x} + L_{sz}\frac{\partial z}{\partial t \partial x} + L_{sp}\frac{\partial p}{\partial t \partial x} + L_{sg}\frac{\partial g}{\partial t \partial x}$$

is the total mixed partial derivative with respect to t and x.

To determine the terms in the Euler - Poisson equation (3.95), we use the integrand from the averaged action (3.92)

$$L = \frac{\hbar^2}{2m}\psi(x,t)\frac{\partial^2\psi(x,t)}{\partial x^2} + \psi(x,t)^2[E-U(x)] + i\hbar\psi(x,t)\frac{\partial\psi(x,t)}{\partial t}. \qquad (3.98)$$

As a result of substituting the Lagrangian (3.98) into expressions (3.96) and (3.97), we obtain

$$L_z = \frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + 2\psi(x,t)[E-U(x)] + i\hbar\frac{\partial\psi(x,t)}{\partial t}; \qquad \frac{\partial^2}{\partial t^2}\{L_t\} = 0;$$

$$\frac{\partial}{\partial x}\{L_p\} = 0; \qquad \frac{\partial^2}{\partial x\partial t}\{L_s\} = 0;$$

$$\frac{\partial}{\partial t}\{L_g\} = 2i\hbar\frac{\partial\psi(x,t)}{\partial t}; \qquad \frac{\partial^2}{\partial x^2}\{L_r\} = 2\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}.$$

(99)

Substituting expressions (3.99) into the Euler - Poisson equation (3.95), we obtain the desired equation for determining the extrema $\psi(x,t)$ of the averaged action functional (3.92)

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \frac{3\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + 2[E-U(x)]\psi(x,t). \qquad (3.100)$$

Generalization to three dimensions reduces to an increase in the number of integrations, and instead of equation (3.100), we obtain

$$i\hbar\frac{\partial\psi(x,y,z,t)}{\partial t} = \frac{3\hbar^2}{2m}\left\{\frac{\partial^2\psi(x,y,z,t)}{\partial x^2} + \frac{\partial^2\psi(x,y,z,t)}{\partial y^2} + \frac{\partial^2\psi(x,y,z,t)}{\partial z^2}\right\} + 2[E-U(x,y,z)]\psi(x,y,z,t), \quad (3.101)$$

or in compact form

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = \frac{3\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + 2[E-U(\vec{r},t)]\psi(\vec{r},t), \qquad (3.102)$$

where **r** is the radius vector with the beginning in the "center" of the investigated object ($r^2 = x^2 + y^2 + z^2$) (see Figure 3.1);

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator.

Equation (3.102) derived in this work is somewhat different from the usual form of the time-dependent Schrödinger equation (3.5)

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + U(\vec{r},t)\,\psi(\vec{r},t). \quad (3.103)$$

But in the case where the wave function does not depend on time [i.e. when $\psi(x,t) = \psi(x)$], equation (3.102) takes the form

$$-\frac{3}{2}\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + U(\vec{r})\,\psi(\vec{r}) = E\,\psi(\vec{r}), \quad (3.104)$$

which almost completely (up to a factor of 3/2 = 1.5) coincides with the time-independent Schrödinger equation for a similar case

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + U(\vec{r})\,\psi(\vec{r}) = E\,\psi(\vec{r})\,.$$

In addition, taking into account expression (3.6), equation (3.102) can be represented as

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = \frac{3\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + 2T(\vec{r},t)\,\psi(\vec{r},t). \quad (3.105)$$

This equation is even more consistent with the Schrödinger equation (3.103), but in this case the potential energy of the particle ("point") $U(\mathbf{r},t)$ is replaced by twice of the kinetic energy $T(\mathbf{r},t)$.

Thus, in this work, Equations (3.102) and (3.104) are derived based on a detailed analysis of the chaotic behavior of a particle ("point"), wandering so that its total mechanical energy E always remains constant (E = const). The results almost completely coincide with the corresponding Schrödinger equations.

Now we divide the total mechanical energy E and the potential energy $U(\mathbf{r}, t)$ of the particle ("points") by its mass m:

$$\varepsilon = E/m \quad (3.106)$$

- we call this massless quantity "energiality";

$$\nu(\mathbf{r},t) = U(\mathbf{r},t)/m \quad (3.107)$$

- we call this massless quantity "potentiality".

Taking into account (3.106) and (3.107), equations (3.102) and (3.104) can be represented as

$$i\hbar\frac{\partial\psi(\bar{r},t)}{\partial t}=-\frac{3\hbar^2}{2m}\nabla^2\psi(\bar{r},t)+\ 2m[\varepsilon-\nu(\vec{r},t)]\psi(\vec{r},t), \tag{3.108}$$

$$-\frac{3\hbar^2}{4m}\nabla^2\psi(\bar{r})+\ m[\varepsilon-\nu(\vec{r})]\psi(\vec{r})=0\,, \tag{3.109}$$

Divide both sides of these equations by $\hbar$

$$i\frac{\partial\psi(\bar{r},t)}{\partial t}=-\frac{3\hbar}{2m}\nabla^2\psi(\bar{r},t)+\ \frac{2m}{\hbar}[\varepsilon-\nu(\vec{r},t)]\psi(\vec{r},t), \tag{3.110}$$

$$-\frac{3\hbar}{4m}\nabla^2\psi(\bar{r})+\ \frac{m}{\hbar}[\varepsilon-\nu(\vec{r})]\psi(\vec{r})=0\,. \tag{3.111}$$

Now we take into account that according to (3.61)

$$\frac{\hbar}{m}=\eta_{par}\,. \tag{3.112}$$

In this case, equations (3.110) and (3.111) can be represented as

$$i\frac{\partial\psi(\bar{r},t)}{\partial t}=-\frac{3}{2}\eta_{par}\nabla^2\psi(\bar{r},t)+\ \frac{2}{\eta_{par}}[\varepsilon-\nu(\vec{r},t)]\psi(\vec{r},t)\,, \tag{3.113}$$

$$-\frac{3}{4}\eta_{par}\nabla^2\psi(\bar{r})+\ \frac{1}{\eta_{par}}[\varepsilon-\nu(\vec{r})]\psi(\vec{r})=0, \tag{3.114}$$

where
$$\eta_{par}=\frac{2\sigma^2_{par\ r}}{\tau_{par\ r}}=\frac{\hbar}{m} \tag{3.115}$$

depends only on the averaged characteristics of the investigated stationary stochastic process:

$$\sigma_{par\ r}=\frac{1}{3}\sqrt{\sigma^2_{par\ x}+\sigma^2_{par\ y}+\sigma^2_{par\ z}} \tag{3.116}$$

is the standard deviation of a randomly moving particle ("point") from the "center" (Figure 3.1);

$$\tau_{par\ r}=\frac{1}{3}\left(\tau_{par\ x}+\tau_{par\ y}+\tau_{par\ z}\right) \tag{3.117}$$

is the average correlation time (more precisely, autocorrelation) of the stochastic process under consideration.

The averaged values of $\sigma_{par\ r}$ and $\tau_{par\ r}$ are geometric in nature and can be empirically estimated, unlike the mass of a particle ("point") m, which in many cases cannot be measured at all.

Equations (3.113) and (3.114) will be called the «generalized Schrödinger equations», since they are suitable for describing the most probable states of point objects of both the microcosm and the macrocosm provided that the stochastic (random) process under consideration is stationary and its total mechanical energy is constant.

3.6 Summary and Conclusions

The article considers a stationary random (stochastic) process associated with the chaotic motion of a particle in the vicinity of the conditionally selected center of a given stochastic system (see Figure 3.1). Based on a detailed consideration of this process, we obtained equation (3.104) [or (3.114)] for the extrema of the averaged functional of the action of a randomly wandering particle, which almost coincided with the time-independent Schrödinger equation (3.105).

Equation (3.114) equally well describes the discrete sets of the averaged behavior of an electron behavior in a potential well, a nucleus in the cytoplasm of a biological cell, the center of the embryonic mass in the womb, a nucleus in the bowels of the planet, a fly in a bank, etc. All these stable stochastic processes have the possibility of transition from one stationary standing in another stationary standing with the absorption or release of a specific portion of the total mechanical "energiality".

In this way, together with the derivation of the generalized Schrödinger equation (3.114), we come to the realization that quantum transitions are inherent not only to objects of atomic scale, but also manifest themselves at all levels of the organization of being.

This is easy to verify, for example, by referring to a fly constantly flying in a large glass jar. With the help of a video camera, you can record its chaotic movements for a long time. If you then scroll through the video at a very high speed, you will not see a fly on the screen, but a stable blurry spot that reflects the probability density of the location of the fly. It should be expected that if the fly is not disturbed by anything, then the blurry spot will resemble a Gaussian probability distribution density with the greatest black in the center of the glass jar. However, if the fly is somehow influenced by an energy transfer, for example, by heat or by ultrasound with a certain frequency, then the average behavior of the fly can change dramatically (in discrete steps). In this case, a blurry spot can change the configuration to an averaged ring or an averaged eight, etc. (Of course, no fly should be made to suffer from such experiments.) The embryo in the womb, the core in the bowels of the planet, and many other similar objects will behave in the same way for long periods of time. It is precisely these different probabilistic configurations with different energy levels that are described by the generalized Schrödinger equation (3.114) derived in this article.

The center of the embryonic mass in the womb, and the core in the bowels of the planet, and many other similar objects whose behavior is studied over fairly long periods of time, will behave in the same way.

The approach proposed in this paper makes it possible to derive the basic equations of nonrelativistic quantum physics (3.113) and (3.114) based on principles that are fundamentally different from the ideological foundations of the Copenhagen and many-worlds interpretations of quantum mechanics. (For example, in this article, the wandering particle under study has a chaotic trajectory and specific dimensions.) However, the mathematical apparatus of quantum mechanics, created by great scientists, remained virtually unchanged, but its logical foundations becomes thereby much clearer.

In a similar way, all the basic equations of quantum field theory can be obtained: the Clen-Gordon equation, Dirac equations, Maxwell equations, etc. Their derivation algorithm is similar to the approach given in this article:

1) express the deterministic action of the particle;

2) find the mean of this action,

3) all the averaged terms in the integrand of the averaged action are represented through the probability density functions $\rho(x)$ and/or $\rho(p_x)$;

4) switch all terms of the Lagrangian of the averaged action to a coordinate representation or a momentum representation;

5) determine the equation for the extrema of the resulting functional (averaged action) through methods of the calculus of variations.

The significance of the derivation of the generalized Schrödinger equations (3.113) and (3.114) presented here is as follows:

- ❖ It becomes clear to what phenomena of the micro- and macrocosm this equation relates, what are the boundaries and conditions for its application.
- ❖ There is no need to apply either Heisenberg's "uncertainty principle" and de Broglie's concept of "matter waves", since, for the derivation of equations (3.113) and (3.114), the procedure (3.45) through (3.48) is used, and this procedure is completely analogous to the transition from the coordinate representation to the momentum one, and vice versa. However, this procedure is obtained based only on the analysis of the properties of a stationary stochastic process, without the involvement of the above hypotheses.
- ❖ The ratio $\hbar/m$ (the "reduced Planck constant" divided by the mass) is determined through the variance and correlation time of the investigated stationary stochastic process (3.115). Therefore, the generalized Schrödinger equations (3.113) and (3.114) do not contain the "mass" of the particle m, and because of this it is necessary to introduce an additional dimensional constant – the reduced Planck constant $\hbar$. "Mass" is (according to the author) one of the "darkest" dimensional

quantities of modern physics (see § 1.7.10 in [19] and Chapter 7 in [20]). There is no doubt that in the final theory the concept of "mass" should be absent, and this article is one of the steps towards the eradication of this concept from scientific ideas about nature.

- The volume and trajectory of the wandering particle return to consideration. Together with them, the physics of the microcosm again acquires the usual logical “ground under its feet”.

We hope that if this work is carefully analyzed and accepted by the scientific community, this will not only allow us to calculate the probabilistic outcomes of complex chaotic processes in both the microcosm and the macrocosm, but also analyze the internal essence of these processes. That is, now we can "Think and calculate."

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